MATHEMAGICS Workbook

Unofficial Release

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CHAPTER 1
MENTAL ADDITION

I remember the day in third grade when I discovered that it was easier to add and subtract from left to right than from right to left, which was the way we had all been taught. Suddenly I was able to blurt out the answers to math problems in class well before my classmates put down their pencils. And I didn't even need a pencil! The method was so simple that I performed most calculations in my head. Looking back, I admit I did so as much tc show off as for any mathematical reason. Most kids outgrow such behavior. Those who don't probably become either teachers or magicians.

In this chapter you will learn the left-to-right method of doing mental addition for numbers that range in size from two to four digits. These mental skills are not only important for doing the tricks in this book but are also indispensable in school or at work, or any time you use numbers. Soon you will be able to retire your calculator and use the full capacity of your mind as you add, subtract, multiply, and divide 2-digit, 3-digit, and even 4-digit numbers.

LEFT-TO-RIGHT ADDITION

There are many good reasons why adding left to right is a superior method for mental calculation. For one thing, you do not have to reverse the numbers (as you do when adding right to left). And if you want to estimate your answer, then adding only the leading digits will get you pretty close. If you are used to working from right to left on paper, it may seem unnatural to add and multiply from left to right. But with practice you will find that it is the most natural and efficient way to do mental calculations.

With the first set of problems—2 digit addition—the left to right method may not seem so advantageous. But be patient. If you stick with me, you will see that the only easy way to solve 3-digit and larger addition problems, all subtraction problems, and most definitely all multiplication and division problems, is from left to right. The sooner you get accustomed to computing this way, the better.

2-DIGIT ADDITION

Our assumption in this chapter is that you know how to add and subtract 1-digit numbers. We will begin with 2-digit addition, something I suspect you can already do fairly well in your head. The following exercises are good practice, however, because you will use the 2-digit addition skills you polish here for larger addition problems, as well as virtually all multiplication problems in later chapters. It also illustrates a fundamental principle of mental arithmetic—namely, to simplify your problem by breaking it into smaller, more manageable components to success—simplify, simplify, simplify.

The easiest 2-digit addition problems, of course, are those that do not require you to cany any numbers. For example:

$$\begin{array}{l} \ \ 47 \\ \underline{+32}\ (30 + 2) \end{array}$$

To add 32 to 47, you can simplify by treating 32 as 30 + 2, add 30 to 47 and then add 2. In this way the problem becomes 77 + 2, which equals 79.

$$\begin{array}{r} 47 \\ \underline{+\ 32} \end{array} \quad \rightarrow \quad \begin{array}{r} 77 \\ \underline{+2} \\ 79 \end{array}$$

Keep in mind that the above diagram is simply a way of representing the mental processes involved in arriving at an answer using one method. While you need to be able to read and understand such diagrams as you work your way through this book, our method does not require you to write down anything yourself.

Now let's try a calculation that requires you to carry a number:

$$\begin{array}{l} \ \ 67 \\ \underline{+28}\ (20 + 8) \end{array}$$

Adding from left to right, you can simplify the problem by adding 67 + 20 = 87; then 87 + 8 = 95.

$$\begin{array}{r} 67 \\ \underline{+\ 28} \end{array} \quad \rightarrow \quad \begin{array}{r} 87 \\ \underline{+8} \\ 95 \end{array}$$

Now try one on your own, mentally calculating from left to right, and then check below to see how we did it:

$$\begin{array}{l} \ \ 84 \\ \underline{+57}\ (50 + 7) \end{array}$$

No problem, right? You added 84 + 50 = 134 and added 134 + 7 = 141.

$$\begin{array}{r} 84 \\ \underline{+57} \end{array} \quad \rightarrow \quad \begin{array}{r} 134 \\ \underline{+\ 7} \\ 141 \end{array}$$

If carrying numbers trips you up a bit, don't worry about it. This is probably the first time you have ever made a systematic attempt at mental calculation, and if you're like most people, it will take you time to get used to it. With practice, however, you will begin to see and hear these numbers in your mind, and carrying numbers when you add will come automatically. Try another problem for practice, again computing it in your mind first, and then checking how we did it:

$$\begin{array}{r} 68 \\ \underline{+45} \end{array} \ (40 + 5)$$

You should have added 68 + 40 = 108, and then 108 + 5 = 113, the final answer. No sweat, right? If you would like to try your hand at more 2-digit addition problems, check out the set of exercises below. (The answers and computations are at the end of the book.)

Exercises: 2-Digit Addition

1)	2)	3)	4)	5)
23	64	95	34	89
+16	+43	+32	+26	+78

6)	7)	8)	9)	10)
73	47	19	55	39
+58	+36	+17	+49	+38

3-DIGIT ADDITION

The strategy for adding 3-digit numbers is the same as for adding 2-digit numbers: you add left to right. After each step, you arrive at a new (and smaller) addition problem. Let's try the following:

$$\begin{array}{r} 538 \\ \underline{+327} \end{array} \ (300 + 20 + 7)$$

After adding the hundreds digit of the second number to the first number (538 + 300 = 838), the problem becomes 838 + 27. Next add the tens digit (838 + 20 = 858), simplifying the problem to 858 + 7 = 865. This thought process can be diagrammed as follows:

$$\begin{array}{r} 538 \\ \underline{+327} \end{array} \rightarrow \begin{array}{r} 838 \\ \underline{+\ 27} \end{array} \rightarrow \begin{array}{r} 858 \\ \underline{+\ 7} \\ 865 \end{array}$$

All mental addition problems can be worked using this method. The goal is to keep simplifying the problem until you are left adding a 1-digit number. It is important to reduce the number of digits you are manipulating because human short-term memory is limited to about 7 digits. Notice that 538 + 327 requires you to hold on to 6 digits in your head, whereas 838 + 27 and 858 + 7 require only 5 and 4 digits, respectively. As you simplify the problems, the problems get easier!

Try the following addition problem in your mind before looking to see how we did it:

$$\begin{array}{r} 623 \\ \underline{+159} \end{array} \ (100 + 50 + 9)$$

Did you reduce and simplify the problem by adding left to right? After adding the hundreds digit (623 + 100 = 723), you were left with 723 + 59. Next you should have added the tens digit (723 + 50 = 773), simplifying the problem to 773 + 9, which you easily summed to 782. Diagrammed, the problem looks like this:

$$\begin{array}{r} 623 \\ \underline{+159} \end{array} \rightarrow \begin{array}{r} 723 \\ \underline{+\ 59} \end{array} \rightarrow \begin{array}{r} 773 \\ \underline{+\ 9} \\ 782 \end{array}$$

When I do these problems mentally, I do not try to *see* the numbers in my mind—I try to *hear* them. I hear the problem 623 + 159 as six **hundred** twenty-three plus one **hundred** fifty-nine; by emphasizing the word "hundred" to myself, I know where to begin adding. Six plus one equals seven, so my next problem is seven **hundred** and twenty-three plus fifty-nine, and so on. When first doing these problems, practice them out loud. Reinforcing yourself verbally will help you learn the mental method much more quickly.

Addition problems really do not get much harder than the following:

$$\begin{array}{r} 858 \\ \underline{+634} \end{array}$$

Now look to see how we did it, below:

$$\begin{array}{r} 858 \\ \underline{+634} \end{array} \rightarrow \begin{array}{r} 1458 \\ \underline{+\ 34} \end{array} \rightarrow \begin{array}{r} 1488 \\ \underline{+\ \ 4} \\ 1492 \end{array}$$

At each step I hear (not see) a "new" addition problem. In my mind the problem sounds like this:

858 plus 634 is 1458 plus 34 is 1488 plus 4 is 1492

Your mind-talk may not sound exactly like mine, but whatever it is you say to yourself, the point is to reinforce the numbers along the way so that you don't forget where you are and have to start the addition problem over again.

Let's try another one for practice:

$$\begin{array}{r} 759 \\ \underline{+496} \end{array} (400 + 90 + 6)$$

This addition problem is a little more difficult than the last one since it requires you to carry numbers in all three steps. However, with this particular problem you have the option of using an alternative method. I am sure you will agree that it is a lot easier to add 500 to 759 that it is to add 496, so try adding 500 and then subtracting the difference:

$$\begin{array}{r} 759 \\ \underline{+496} \end{array} (500 - 4)$$

$$\begin{array}{r} 759 \\ \underline{+496} \end{array} \rightarrow \begin{array}{r} 1259 \\ \underline{-\ \ \ 4} \\ 1255 \end{array}$$

So far, you have consistently broken up the second number in any problem to add to the first. It really does not matter which number you choose to break up as long as you are consistent. That way, your mind will never have to waste time deciding which way to go. If the second number happens to be a lot simpler than the first, I switch them around, as in the following example:

$$\begin{array}{r} 207 \\ +\underline{528} \end{array} \rightarrow \begin{array}{r} 528 \\ +\underline{207} \end{array} \rightarrow \begin{array}{r} 728 \\ +\underline{\ \ 7} \\ 735 \end{array}$$

Let's finish up by adding 3-digit to 4-digit numbers. Again, since most human memory can only hold about seven digits at a time, this is about as large a problem as you can handle without resorting to artificial memory devices. Often (especially within multiplication problems) one or both of the numbers will end in 0, so we shall emphasize those types of problems. We begin with an easy one:

$$\begin{array}{r} 2700 \\ +\underline{567} \end{array}$$

Since 27 *hundred* + 5 *hundred* is 32 *hundred*, we simply attach the 67 to get 32 hundred and 67, or 3267. The process is the same for the following problems:

$$\begin{array}{r} 3240 \\ +\underline{\ \ 18} \end{array} \qquad \begin{array}{r} 3240 \\ +\underline{\ \ 72} \end{array}$$

Because 40 + 18 = 58, the first answer is 3258. For the second problem, since 40 + 72 exceeds 100, you know the answer will be 33 hundred and *something.* Because 40 + 72 = 112, you end up with 3312.

These problems are easy because the digits only overlap in one place, and hence can be solved in a single step. Where digits overlap in two places, you require two steps. For instance:

$$\begin{array}{r} 4560 \\ +\underline{171} \end{array} \ (100 + 71)$$

This problem requires two steps, as diagrammed the following way:

$$\begin{array}{r} 4560 \\ +\underline{171} \end{array} \rightarrow \begin{array}{r} 4660 \\ +\underline{\ \ 71} \\ 4731 \end{array}$$

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Practice the following 3-digit addition exercises, and then add some of your own if you like (pun intended!) until you are comfortable doing them mentally without having to look down at the page.

Exercises: 3-Digit Addition

1)	2)	3)	4)	5)
242	312	635	457	912
+137	+256	+814	+241	+475
6)	7)	8)	9)	10)
852	457	878	276	877
+378	+269	+797	+689	+539
11)	12)	13)	14)	15)
5400	1800	6120	7830	4240
+252	+855	+136	+348	+371

Answers can be found in the back of the book.

CHAPTER 2
LEFT-TO-RIGHT SUBTRACTION

For most of us, it is easier to add than to subtract. But if you continue to compute from left to right and to break problems down into simpler components (using the principle of simplification, as always), subtraction can become almost as easy as addition.

2-DIGIT SUBTRACTION

For 2-digit subtraction, as in addition, your goal is to keep simplifying the problem until you are reduced to subtracting a 1-digit number. Let's begin with a very simple subtraction problem:

$$\begin{array}{rl} 86 & \\ \underline{-\ 25} & (20 + 5) \end{array}$$

After each step, you arrive at a new and smaller subtraction problem. Begin by subtracting 86 - 20 = 66. Your problem becomes 66 - 5 = 61, your final answer. The problem can be diagrammed this way:

$$\begin{array}{r} 86 \\ \underline{-\ 25} \end{array} \rightarrow \begin{array}{r} 66 \\ \underline{-\ 5} \\ 61 \end{array}$$

Of course, subtraction problems are considerably easier when they do not involve borrowing. When they do, there are a number of strategies you can use to make them easier. For example:

$$\begin{array}{rl} 86 & \\ \underline{-\ 29} & (20 + 9) \text{ or } (30 - 1) \end{array}$$

There are two different ways to solve this problem mentally:

1. You can simplify the problem by breaking down 29 into 20 and 9, subtracting 20, then subtracting 9:

$$\begin{array}{r} 86 \\ \underline{-\ 29} \end{array} \rightarrow \begin{array}{r} 66 \\ \underline{-\ 9} \\ 57 \end{array}$$

2. You can treat 29 as 30 - 1, subtracting 30, then adding back 1:

$$\begin{array}{r} 86 \\ \underline{-\ 29} \end{array} \rightarrow \begin{array}{r} 56 \\ \underline{+1} \\ 57 \end{array}$$

Here is the rule for deciding which method to use:

If the subtraction of two numbers requires you to borrow a number, round up the second number to a multiple of ten and add back the difference.

For example,

$$\begin{array}{r} 54 \\ \underline{-\ 28}\ (30 - 2) \end{array}$$

Since this problem requires you to borrow, round up 28 to 30, subtract, and then add back 2 to get 26 as your final answer:

$$\begin{array}{r} 54 \\ \underline{-\ 28} \end{array} \rightarrow \begin{array}{r} 24 \\ \underline{+2} \\ 26 \end{array}$$

Now try your hand at this 2-digit subtraction problem:

$$\begin{array}{r} 81 \\ \underline{-\ 37} \end{array}$$

Easy, right? You just round up 37 to 40, subtract 40 from 81, which gives you 41, and then add back the difference of 3 to arrive at 44, the final answer:

$$\begin{array}{r} 81 \\ \underline{-\ 37} \end{array} \rightarrow \begin{array}{r} 41 \\ \underline{+3} \\ 44 \end{array}$$

In a short time you will become comfortable working subtraction problems both ways. Just use the rule above to decide which method will work best.

Exercises: 2-Digit Subtraction

1)	2)	3)	4)	5)
38	84	92	67	79
- 23	- 59	- 34	- 48	- 29
6)	7)	8)	9)	10)
63	51	89	125	148
- 46	- 27	- 48	- 79	- 86

3-DIGIT SUBTRACTION

Now let's try a 3-digit subtraction problem:

$$\begin{array}{r} 958 \\ \underline{-\ 417} \end{array} \ (400 + 10 + 7)$$

This particular problem does not require you to borrow any numbers, so you should not find it too hard. Simply subtract one digit at a time, simplifying as you go.

$$\begin{array}{r} 958 \\ \underline{-\ 417} \end{array} \rightarrow \begin{array}{r} 558 \\ \underline{-\ 17} \end{array} \rightarrow \begin{array}{r} 548 \\ \underline{-\ \ 7} \\ 541 \end{array}$$

Now let's look at a 3-digit subtraction problem that requires you to borrow a number:

$$\begin{array}{r} 747 \\ \underline{-\ 598} \end{array} \ (600 - 2)$$

At first glance this probably looks like a pretty tough problem, but if you round up by 2, subtract 747 - 600 = 147, then add back 2, you reach your final answer of 147 + 2 = 149.

$$\begin{array}{r} 747 \\ \underline{-\ 598} \end{array} \rightarrow \begin{array}{r} 147 \\ \underline{+\ 2} \\ 149 \end{array}$$

Now try one yourself:

$$\begin{array}{r} 853 \\ \underline{-\ 692} \end{array}$$

Did you round 692 up to 700 and then subtract 700 from 853? If you did, you got 853 - 700 = 153. Since you subtracted by 8 too much, did you add back 8 to reach 161, the final answer?

$$\begin{array}{r} 853 \\ \underline{-\ 692} \end{array} \rightarrow \begin{array}{r} 153 \\ \underline{+\ \ 8} \\ 161 \end{array}$$

Now, I admit we have been making life easier for you by choosing numbers you don't have to round up by much. But what happens when it isn't so easy to figure out how much to add back when you have subtracted too much? The following 3-digit subtraction problem illustrates exactly what I mean:

$$\begin{array}{l} \ \ 725 \\ \underline{-\ 468}\ (400 + 60 + 8) \text{ or } (500 - ??) \end{array}$$

If you subtract one digit at a time, simplifying as you go, your sequence will look like this:

$$\begin{array}{r} 725 \\ \underline{-\ 468} \end{array} \rightarrow \begin{array}{r} 325 \\ \underline{-\ 68} \end{array} \rightarrow \begin{array}{r} 265 \\ \underline{-\ \ 8} \\ 257 \end{array}$$

What happens if you round up to 500?

$$\begin{array}{r} 725 \\ \underline{-\ 468} \end{array} \rightarrow \begin{array}{r} 225 \\ \underline{+??} \end{array}$$

Subtracting 500 is easy: 725 - 500 = 225. But you have subtracted too much. The trick is to figure out exactly how much.

At first glance, the answer is far from obvious. To find it, you need to know how far 468 is from 500. The answer can be found by using "complements," a nifty technique that will make many 3-digit subtraction problems a lot easier to figure out.

USING COMPLEMENTS (YOU'RE WELCOME!)

Quick, how far from 100 are each of these numbers?

57	68	49	21	79

Here are the answers:

$$\begin{array}{r}57\\ \underline{+43}\\ 100\end{array}\qquad \begin{array}{r}68\\ \underline{+32}\\ 100\end{array}\qquad \begin{array}{r}49\\ \underline{+51}\\ 100\end{array}\qquad \begin{array}{r}21\\ \underline{+79}\\ 100\end{array}\qquad \begin{array}{r}79\\ \underline{+21}\\ 100\end{array}$$

Notice that for each pair of numbers that add to 100, the first digits (on the left) add to 9 and the last (on the right) add to 10. We say that 43 is the complement of 57, 32 the complement of 68, and so on.

Now you find the complement of these 2-digit numbers:

37	59	93	44	08

To find the complement of 37, first figure out what you need to add to 3 to get 9. (The answer is 6.) Then figure out what you need to add to 7 to get 10. (The answer is 3.) Hence, 63 is your complement.

The other complements are 41, 7, 56, 92. Notice that, like everything else you do as a mathemagician, the complements are determined from left to right. As we have seen, the first digits add to 9, and the second digits add to 10. (An exception occurs in numbers ending in 0—e.g., 30 + 70 = 100—but those complements are simple!)

What do complements have to do with mental subtraction? Well, they allow you to convert difficult subtraction problems into straightforward addition problems. Let's consider the last subtraction problem that gave us some trouble:

$$\begin{array}{rl}725 & \\ \underline{-\ 468} & (500 - 32)\end{array}$$

To begin, you subtracted 500 instead of 468 to arrive at 225 (725 - 500 = 225). But then, having subtracted too much, you needed to figure out how much too much. Using complements gives you the answer in a flash. How far is 468 from 500? The same distance as 68 is from 100. If you take the complement of 68 the way we have shown you, you will arrive at 32. Add 32 to 225 and you will arrive at 257, your final answer.

$$\begin{array}{r} 725 \\ \underline{-\ 468} \\ \end{array} \rightarrow \begin{array}{r} 225 \\ \underline{+\ 32} \\ 257 \end{array}$$

Try another 3-digit subtraction problem:

$$\begin{array}{l} \ \ 821 \\ \underline{-\ 259}\ (300 - 41) \end{array}$$

To compute this mentally, subtract 300 from 821 to arrive at 521, then add back the complement of 59, which is 41, to arrive at 562, our final answer. The procedure looks like this:

$$\begin{array}{r} 821 \\ \underline{-\ 259} \\ \end{array} \rightarrow \begin{array}{r} 521 \\ \underline{+\ 41} \\ 562 \end{array}$$

Here is another problem for you to try:

$$\begin{array}{l} \ \ 645 \\ \underline{-\ 372}\ (400 - 28) \end{array}$$

Check your answer and the procedure for solving the problem below:

$$\begin{array}{r} 645 \\ \underline{-\ 372} \\ \end{array} \rightarrow \begin{array}{r} 245 \\ \underline{+\ 28} \\ \end{array} \rightarrow \begin{array}{r} 265 \\ \underline{+\ \ 8} \\ 273 \end{array}$$

Subtracting a 3-digit number from a 4-digit number is not much harder, as the next example illustrates:

$$\begin{array}{l} 1246 \\ \underline{-\ 579}\ (600 - 21) \end{array}$$

By rounding up you subtract 600 from 1246, leaving 646, then add back the complement of 79, which is 21. Your final answer is 646 + 21 = 667.

$$\begin{array}{r} 1246 \\ \underline{-\ 579} \end{array} \rightarrow \begin{array}{r} 646 \\ \underline{+\ 21} \\ 667 \end{array}$$

Try the 3-digit subtraction exercises below, and then create more of your own for additional (or should that be subtractional?) practice.

Exercises: 3-Digit Subtraction

1)	2)	3)	4)	5)
583	936	587	763	204
- 271	- 725	- 298	- 486	- 185
6)	7)	8)	9)	10)
793	219	978	455	772
- 402	- 176	- 784	- 319	- 596
11)	12)	13)	14)	15)
873	564	1428	2345	1776
- 357	- 228	- 571	- 571	- 987

Answers can be found in the back of the book.

CHAPTER 3
BASIC MULTIPLICATION

I spent practically my entire childhood devising faster and faster ways to perform mental multiplication; I was diagnosed as hyperactive and my parents were told that I had a short attention span and probably would not be successful in school. Ironically, it was my short attention span that motivated me to develop quick ways to do arithmetic. I could not possibly sit still long enough to carry out math problems with pencil and paper. Once you have mastered the techniques described in this chapter, you won't want to rely on pencil and paper again, either.

In this chapter you will learn how to multiply 1-digit numbers by 2-digit numbers and 3-digit numbers in your head. You will also learn a phenomenally fast way to square 2-digit numbers. Even friends with calculators won't be able to keep up with you Believe me, virtually everyone will be dumbfounded by the fact that such problems can not only be done mentally, but can be computed so quickly. I sometimes wonder whether we were not cheated in school; these methods are so simple once you learn them.

There is one small prerequisite for mastering the mathemagic tricks in this chapter—you need to know the multiplication tables through 10. In fact, to really make headway, you need to know your multiplication tables backward and forward. For those of you who need to shake the cobwebs loose, consult the figure below.

X	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Once you've got your tables down, you are in for some fun, because multiplication provides ample opportunities for creative problem solving.

2-BY-l MULTIPLICATION PROBLEMS

If you worked your way through the last chapters, you got into the habit of adding and subtracting from left to right. You will do virtually all the calculations in this chapter from left to right, as well. This is undoubtedly the opposite of what you learned in school. But you'll soon see how much easier it is to think from left to right than from right to left. (For one thing, you can start to say your answer aloud before you have finished the calculation. That way you seem to be calculating even faster than you are!)

Let's tackle our first problem:

$$\begin{array}{r} 42 \\ \underline{\times 7} \end{array}$$

First, multiply 40 x 7 = 280. Next, multiply 2 x 7 = 14, add 14 to 280 (left to right, of course) to arrive at 294, the correct answer. We illustrate this procedure below. We have omitted diagramming the mental addition of 280 + 14, since you learned in Chapter 1 how to do this computation.

$$\begin{array}{rl} & 42\ (40 + 2) \\ & \underline{\times 7} \\ 40 \times 7 = & 280 \\ 2 \times 7 = & \underline{+14} \\ & 294 \end{array}$$

At first you will need to look down at the problem while calculating it to recall the next operation. With practice you will be able to forgo this step and compute the whole thing in your mind.

Let's try another example:

$$\begin{array}{r} 48 \\ \underline{\times 4} \end{array}$$

Your first step is to break down the problem into small multiplication tasks that you can perform mentally with ease. Since 48 = 40 + 8, multiply 40 x 4 = 160, then add 8 x 4 = 32. The answer is 192. (Note: If you are wondering *why* this process works, see the section "Why These Tricks Work" at the end of the chapter.)

$$\begin{array}{rl} & 48\ (40 + 8) \\ & \underline{\times\ 4} \\ 40 \times 4 = & 160 \\ 8 \times 4 = & \underline{+32} \\ & 192 \end{array}$$

Here are two more mental multiplication problems that you should be able to solve fairly quickly. Try calculating them in your head before looking at how we did it.

$$\begin{array}{rl} & 62\ (60 + 2) \\ & \underline{\times\ 3} \\ 60 \times 3 = & 180 \\ 2 \times 3 = & \underline{+\ 6} \\ & 186 \end{array} \qquad \begin{array}{rl} & 71\ (70 + 1) \\ & \underline{\times\ 9} \\ 70 \times 9 = & 630 \\ 1 \times 9 = & \underline{+\ 9} \\ & 639 \end{array}$$

These two examples are especially simple because they do not require you to carry any numbers. Another especially easy type of mental multiplication problem involves numbers that begin with 5. When the 5 is multiplied by an even digit, the first product will be a multiple of 100, which makes the resulting addition problem a snap:

$$\begin{array}{rl} & 58\ (50 + 8) \\ & \underline{\times\ 4} \\ 50 \times 4 = & 200 \\ 8 \times 4 = & \underline{+32} \\ & 232 \end{array}$$

Try your hand at the following problem:

$$\begin{array}{rl} & 87\ (80 + 7) \\ & \underline{\times\ 5} \\ 80 \times 5 = & 400 \\ 7 \times 5 = & \underline{+35} \\ & 435 \end{array}$$

Notice how much easier this problem is to do from left to right. It takes far less time to calculate "400 plus 35" mentally than it does to apply the pencil-and-paper method of "putting down the 5 and carrying the 3."

The following two problems are harder because they force you to carry numbers when you come to the addition:

$$\begin{array}{rl} & 38\ (30 + 8) \\ & \underline{\times\ 9} \\ 30 \times 9 = & 270 \\ 8 \times 9 = & \underline{+72} \\ & 342 \end{array} \qquad \begin{array}{rl} & 67\ (60 + 7) \\ & \underline{\times\ 8} \\ 60 \times 8 = & 480 \\ 7 \times 8 = & \underline{+56} \\ & 536 \end{array}$$

As usual, break these problems down into easier problems. For the one on the left, multiply 30 x 9 plus 8 x 9, giving you 270 + 72. The addition problem is slightly harder because it involves carrying a number. Here 270 + 70 + 2 = 340 + 2 = 342.

With practice, you will become more adept at juggling problems like these in your head, and those that require you to carry numbers will be almost as easy as the others.

Rounding Up

You saw in the last chapter how useful rounding up can be when it comes to subtraction. The same goes for multiplication, especially when the numbers you are multiplying end in an 8 or 9.

Let's take the problem of 69 x 6, illustrated below. On the left we have calculated it the usual way, by adding. On the right, however, we have rounded 69 up to 70, which is an easy number to multiply, and for many people, it is easier to subtract 420 - 6 than it is to add 360 + 54 when mentally calculating.

		or		
	69 (60 + 9)			69 (70 - 1)
	x 6			x 6
60 x 6 =	360		70 x 6 =	420
9 x 6 =	+54		- 1 x 6 =	- 6
	414			414

The following example also shows how much easier rounding up can be:

		or		
	78 (70 + 8)			78 (80 - 2)
	x 9			x 9
70 x 9 =	630		80 x 9 =	720
8 x 9 =	+72		- 2 x 9 =	- 18
	702			702

The subtraction method works especially well for numbers just one or two digits away from a multiple of 10. It does not work so well when you need to round up more than two digits because the subtraction portion of the problem gets out of hand. As it is, you may prefer to stick with the addition method. Personally, I only use the addition method because in the time spent deciding which method to use, I could have already done the calculation!

So that you can perfect your technique I strongly recommend practicing more 2-by-1 multiplication problems. Below are 20 problems for you to tackle. I have supplied you with the answers in the back, including a breakdown of each component of the multiplication. If, after you've worked out these problems, you would like to practice more, make up your own. Calculate mentally, then check your answer with a calculator. Once you feel confident that you can perform these problems rapidly in your head, you are ready to move to the next level of mental calculation.

Exercises: 2-by-l Multiplication

1) 82 x9	2) 43 x7	3) 67 x5	4) 71 x3	5) 93 x8
6) 49 x9	7) 28 x4	8) 53 x5	9) 84 x5	10) 58 x6
11) 97 x4	12) 78 x2	13) 96 x9	14) 75 x4	15) 57 x7
16) 37 x6	17) 46 x2	18) 76 x8	19) 29 x3	20) 64 x8

3-BY-l MULTIPLICATION PROBLEMS

Now that you know how to do 2-by-l multiplication problems in your head, you will find that multiplying three digits by a single digit is not much more difficult. You can get started with the following 3-by-l problem (which is really just a 2-by-l problem in disguise):

$$\begin{array}{rr} & 320\ (300 + 20) \\ & \underline{\times\ 7} \\ 300 \times 7 = & 2100 \\ 20 \times 7 = & \underline{+140} \\ & 2240 \end{array}$$

What could be easier? Let's try another 3-by-l problem similar to the one you just did, except we have replaced the 0 with a 6 so that you have another step to perform:

$$
\begin{array}{rl}
 & 326\ (300 + 20 + 6) \\
 & \underline{\times\ 7} \\
300 \times 7 = & 2100 \\
20 \times 7 = & \underline{+140} \\
 & 2240 \\
6 \times 7 = & \underline{+\ \ 42} \\
 & 2282
\end{array}
$$

In this case, you simply add the product of 6 x 7, which you already know to be 42, to the first sum of 2240. Since you do not need to carry any numbers, it is easy to add 42 to 2240 to arrive at the total of 2282.

In solving this and other 3-by-l multiplication problems, the difficult part may be holding in memory the first sum (in this case, 2240) while doing the next multiplication problem (in this case, 6 x 7). There is no magic secret to remembering that first number, but with practice I guarantee you will improve your concentration so that holding on to numbers while performing other functions will get easier.

Let's try another problem:

$$
\begin{array}{rl}
 & 647\ (600 + 40 + 7) \\
 & \underline{\times\ \ \ 4} \\
600 \times 4 = & 2400 \\
40 \times 4 = & \underline{+160} \\
 & 2560 \\
7 \times 4 = & \underline{+28} \\
 & 2588
\end{array}
$$

Even if the numbers are large, the process is just as simple. For example:

$$
\begin{array}{rl}
 & 987\ (900 + 80 + 7) \\
 & \underline{\times\ \ \ 9} \\
900 \times 9 = & 8100 \\
80 \times 9 = & \underline{+720} \\
 & 8820 \\
7 \times 9 = & \underline{+\ \ 63} \\
 & 8883
\end{array}
$$

When first solving these problems, you may have to glance down at the page as you go along to remind yourself what the original problem was. This is okay at first. But try to break the habit so that eventually you are holding the problem entirely in memory.

In the last section on 2-by-l multiplication problems, we saw that problems involving numbers that begin with 5 are sometimes especially easy to solve. The same is true for 3-by-l problems:

$$
\begin{array}{rr}
 & 563\ (500 + 60 + 3) \\
 & \underline{\times\ 6} \\
500 \times 6 = & 3000 \\
60 \times 6 = & 360 \\
3 \times 6 = & \underline{+18} \\
 & 3378
\end{array}
$$

Notice that whenever the first product is a multiple of 1000, the resulting addition problem is no problem at all because you do not have to carry any numbers and the thousands digit does not change. If you were solving the problem above in front of an audience, you would be able to say your first product—"3000..."—out loud with complete confidence that a carried number would not change it to 4000. (As an added bonus, by quickly saying the first digit, it gives the illusion that you computed the answer immediately!) Even if you are practicing alone, saying your first product out loud frees up some memory space while you work on the remaining 2-by-l problem, which you can say out loud as well—in this case, "...three hundred seventy-eight."

Try the same approach in solving the next problem, where the multiplier is a 5:

$$
\begin{array}{rr}
 & 663\ (600 + 60 + 3) \\
 & \underline{\times\ \ 5} \\
600 \times 5 = & 3000 \\
60 \times 5 = & 300 \\
3 \times 5 = & \underline{+\ 15} \\
 & 3315
\end{array}
$$

Because the first two digits of the 3-digit number are even, you can say the answer as you calculate it without having to add anything! Don't you wish all multiplication problems were so easy?

Let's escalate the challenge by trying a couple of problems that require you to carry a number.

$$
\begin{array}{rr}
 & 184\ (100 + 80 + 4) \\
 & \underline{\times\ \ 7} \\
100 \times 7 = & 700 \\
80 \times 7 = & \underline{+560} \\
 & 1260 \\
4 \times 7 = & \underline{+\ 28} \\
 & 1288
\end{array}
$$

$$
\begin{array}{rr}
 & 684\ (600 + 80 + 4) \\
 & \underline{\times\quad 9} \\
600 \times 9 = & 5400 \\
80 \times 9 = & \underline{+720} \\
 & 6120 \\
4 \times 9 = & \underline{+\ \ 36} \\
 & 6156
\end{array}
$$

In the next two problems you need to carry a number at the end of the problem instead of at the beginning:

$$
\begin{array}{rr}
 & 648\ (600 + 40 + 8) \\
 & \underline{\times\quad 9} \\
600 \times 9 = & 5400 \\
40 \times 9 = & \underline{+360} \\
 & 5760 \\
8 \times 9 = & \underline{+\ \ 72} \\
 & 5832
\end{array}
$$

$$
\begin{array}{rr}
 & 376\ (300 + 70 + 6) \\
 & \underline{\times\quad 4} \\
300 \times 4 = & 1200 \\
70 \times 4 = & \underline{+280} \\
 & 1480 \\
6 \times 4 = & \underline{+\ \ 24} \\
 & 1504
\end{array}
$$

The first part of each of these problems is easy enough to compute mentally. The difficult part comes in holding the preliminary answer in your head while computing the final answer. In the case of the first problem, it is easy to add 5400 + 360 = 5760, but you may have to repeat 5760 to yourself several times while you multiply 8 x 9 = 72. Then add 5760 + 72. Sometimes at this stage I will start to say my answer aloud before finishing. Because I know I will have to carry when I add 60 + 72, I know that 5700 will become 5800, so I say "fifty-eight hundred..." Then I pause to compute 60 + 72 = 132. Because I have already carried, I say only the last two digits, "...and thirty-two!" And there is the answer: 5832.

The next two problems require you to carry two numbers each, so they may take you longer than those you have already done. But with practice you will get faster:

$$\begin{array}{rl} & 489\ (400 + 80 + 9) \\ & \underline{\times\quad 7} \\ 400 \times 7 = & 2800 \\ 80 \times 7 = & \underline{+560} \\ & 3360 \\ 9 \times 7 = & \underline{+\ \ 63} \\ & 3423 \end{array}$$

$$\begin{array}{rl} & 224\ (200 + 20 + 4) \\ & \underline{\times\quad 9} \\ 200 \times 9 = & 1800 \\ 20 \times 9 = & \underline{+180} \\ & 1980 \\ 4 \times 9 = & \underline{+\ \ 36} \\ & 2016 \end{array}$$

When you are first tackling these problems, repeat the answers to each part out loud as you compute the rest. In the first problem, for example, start out by saying, "Twenty-eight hundred plus five hundred and sixty" a couple of times out loud to reinforce the two numbers in memory while you add them together. Repeat the answer—"thirty-three hundred and sixty"—several times while you multiply 9 x 7 = 63. Then repeat "thirty-three hundred and sixty plus sixty-three" aloud until you compute the final answer of 3423. If you are thinking fast enough to recognize that adding 60 + 63 will require you to carry a 1, you can begin to give the final answer a split second before you know it—"thirty-four hundred...and twenty-three!"

Let's end this section on 3-by-l multiplication problems with some special problems you can do in a flash because they require one addition step instead of two:

$$\begin{array}{rl} & 511\ (500 + 11) \\ & \underline{\times\quad 7} \\ 500 \times 7 = & 3500 \\ 11 \times 7 = & \underline{+\ \ 77} \\ & 3577 \end{array}$$

```
          925 (900 + 25)
        x   8
900 x 8 = 7200
 25 x 8 = +200
          7400
```

```
          825 (800 + 25)
        x   3
800 x 3 = 2400
 25 x 3 = +  75
          2475
```

In general, if the product of the last two digits of the first number and the multiplier is known to you without having to calculate it (for instance, you may know that 25 x 8 = 200 automatically since 8 quarters equals $2.00), you will get to the final answer much more quickly. For instance, if you know without calculating that 75 x 4 = 300, then it is a breeze to compute 975 x 4:

```
          975 (900 + 75)
        x   4
900 x 4 = 3600
 75 x 4 = +300
          3900
```

To reinforce what you have just learned, solve the following 3-by-l multiplication problems in your head; then check your computations and answers with ours (in the back of the book). I can assure you from experience that doing mental calculations is just like riding a bicycle or typing. It might seem impossible at first, but once you've mastered it, you will never forget how to do it.

Exercises: 3-by-l Multiplication

1) 431 x 6	2) 637 x 5	3) 862 x 4	4) 957 x 6
5) 927 x 7	6) 728 x 2	7) 328 x 6	8) 529 x 9
9) 807 x 9	10) 587 x 4	11) 184 x 7	12) 214 x 8

13) 757 x 8	14) 259 x 7	15) 297 x 8	16) 751 x 9
17) 457 x 7	18) 339 x 8	19) 134 x 8	20) 611 x 3
21) 578 x 9	22) 247 x 5	23) 188 x 6	24) 968 x 6
25) 499 x 9	26) 670 x 4	27) 429 x 3	28) 862 x 5
29) 285 x 6	30) 488 x 9	31) 693 x 6	32) 722 x 9
33) 457 x 9	34) 767 x 3	35) 312 x 9	36) 691 x 3

SQUARING 2-DIGIT NUMBERS

Squaring numbers in your head (multiplying a number by itself) is one of the easiest yet most impressive feats of mental calculation you can do. I can still recall where I was when I discovered how to do it. I was 14, sitting on a bus on the way to visit my father at work in downtown Cleveland. It was a trip I made often, so my mind began to wander. I'm not sure why, but I began thinking about the numbers that add up to 20. How large could the product of two such numbers get?

1 started in the middle with 10 x 10 (or 10^2), the product of which is 100. Next, I multiplied 9 x 11 = 99, 8 x 12 = 96, 7 x 13 = 91, 6 x 14 = 84, 5 x 15 = 75, 4 x 16 = 64, and so on. I noticed that the products were getting smaller, and their difference from 100 was 1,4, 9, 16, 25, 36—or 1^2, 2^2, 3^2, 4^2, 5^2, 6^2 (see figure below)

Numbers that add to 20		Distance from 10	Their Product	Product's difference from 10^2
10	10	0	100	0
9	11	1	99	1
8	12	2	96	4
7	13	3	91	9
6	14	4	84	16
5	15	5	75	25
4	16	6	64	36
3	17	7	51	49
2	18	8	36	64
1	19	9	19	81

I found this pattern astonishing. Next I tried numbers that add to 26 and got similar results. First I worked out 13 =169, then computed 12 x 14 = 168, 11 x 15 = 165, 10 x 16 = 160, 9 x 17 = 153, and so on. Just as before, the distance these products were from 169 was 1^2, 2^2, 3^2, 4^2, and so on (see figure below).

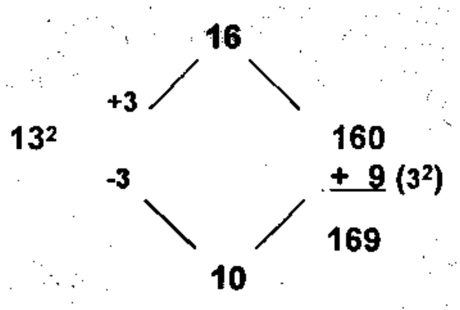
Numbers that add to 26		Distance from 13	Their Product	Product's difference from 13^2
13	13	0	169	0
12	14	1	168	1
11	15	2	165	4
10	16	3	160	9
9	17	4	153	16
8	18	5	144	25

There is actually a simple algebraic explanation for this phenomenon (see the last section of this chapter). At the time, I didn't know my algebra well enough to prove that this pattern would always occur, but I experimented with enough examples to become convinced of it.

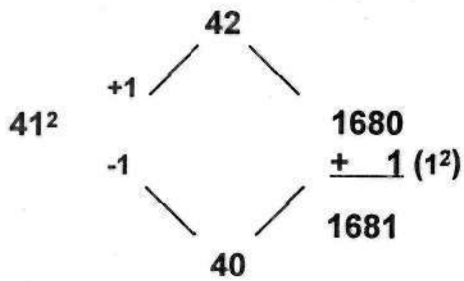
Then I realized this pattern could help me square numbers more easily.

Suppose I wanted to square the number 13, I said to myself. Instead of multiplying 13 x 13, why not get an approximate answer by using two numbers that are easier to multiply but also add up to 26? I chose 10 x 16 = 160. To get an answer, I just added 3 (since 10 and 16 are each 3 away from 13). Since 3 = 9, 13. = 160 + 9 = 169. Neat!

This method is diagrammed as follows:

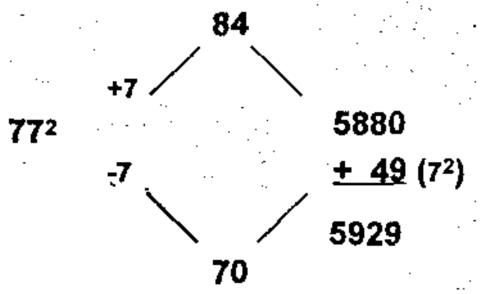


Now let's see how this works for another square:

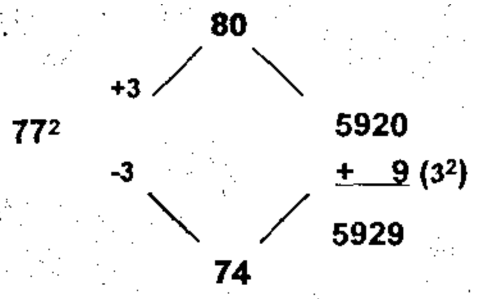


To square 41, subtract 1 to obtain 40 and add 1 to obtain 42. Next multiply 40 x 42. Don't panic! This is simply a 2-by-l multiplication problem (specifically, 4 x 42) in disguise. Since 4 x 42 = 168, 40 x 42 = 1680. Almost done! All you have to add is the square of 1 (the number by which you went up and down from 41), giving you 1680 + 1 = 1681.

Can squaring a 2-digit number be this easy? Yes, with this method and a little practice, it can. And it works whether you initially round down or round up. For example, let's examine 77 , working it out both by rounding up and by rounding down:



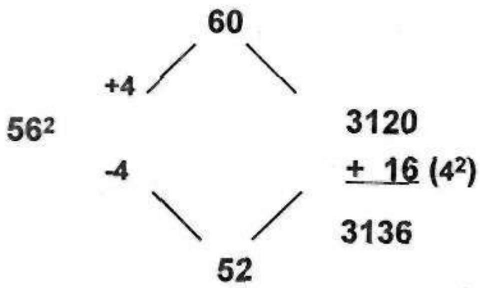
or



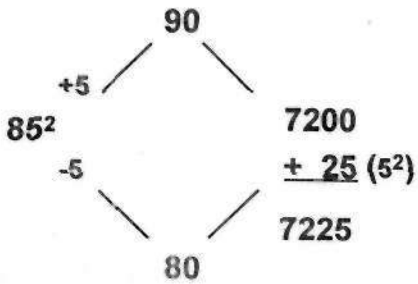
In this instance the advantage of rounding up is that you are virtually done as soon as you have completed the multiplication problem because it is simple to add 9 to a number ending in 0!

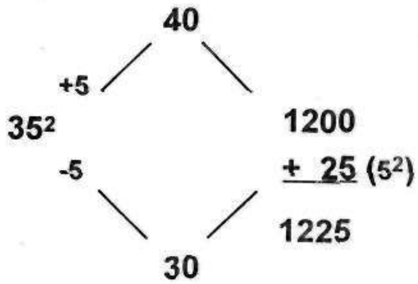
In fact, for all 2-digit squares, I always round up or down to the nearest multiple of 10. So if the number to be squared ends in 6, 7, 8, or 9, round up, and if the number to be squared ends in 1,2, 3, 4, round down. (If the number ends in 5, do both!) With this strategy you will add only the numbers 1, 4, 9, 16, or 25 to your first calculation.

Let's try another problem. Calculate 56 in your head before looking at how we did it, below:



Squaring numbers that end in 5 is even easier. Since you will always round up and down by 5, the numbers to be multiplied will both be multiples of 10. Hence, the multiplication and the addition are especially simple. We have worked out 85 and 35 , below:

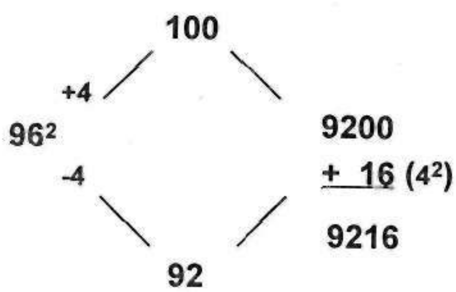




When you are squaring a number that ends in 5, rounding up and down allows you to blurt out the first part of the answer immediately and then finish it with 25. For example, if you want to compute 75 , rounding up to 80 and down to 70 will give you "Fifty-six hundred...and twenty-five!"

For numbers ending in 5, you should have no trouble beating someone with a calculator, and with a little practice with the other squares, it won't be long before you can beat the calculator with any 2-digit square number. Even large numbers are not to be feared. You can ask someone to give you a really big 2-digit number, something in the high 90's, and it will sound as though you've chosen an impossible problem to compute. But, in fact, these are even easier because they allow you to round up to 100.

Let's say your audience gives you 96 . Try it yourself, and then check how we did it.



Wasn't that easy? You should have rounded up by 4 to 100 and down by 4 to 92, and then multiplied 100 x 92 for 9200. At this point you can say out loud, "Ninety-two hundred," and then finish up with "sixteen" and enjoy the applause!

Exercises: 2-Digit Squares

Compute the following:

1) 14^2	2) 27^2	3) 65^2	4) 89^2	5) 98^2
6) 31^2	7) 41^2	8) 59^2	9) 26^2	10) 53^2
11) 21^2	12) 64^2	13) 42^2	14) 55^2	15) 75^2
16) 45^2	17) 84^2	18) 67^2	19) 103^2	20) 208^2

WHY THESE **TRICKS WORK**

This section is presented for teachers, students, math buffs, and anyone curious as to why our methods work. Some people may find the theory as interesting as the application. Fortunately, you need not understand why our methods work in order to understand how to apply them. All magic tricks have a rational explanation behind them, and mathemagic tricks are no different. It is here that the mathemagician reveals his deepest secrets!

In this chapter on multiplication problems, the distributive law is what allows us to break down problems into their component parts. The distributive law states that for any numbers *a*, *b*, and *c*:

$$(b + c) \times a = (b \times a) + (c \times a)$$

That is, the outside term, *a*, is distributed, or separately applied, to each of the inside terms, *b* and *c*. For example, in our first mental multiplication problem of 42 x 7, we arrived at the answer by treating 42 as 40 + 2, then distributing the 7 as follows:

$$42 \times 7 = (40 + 2) \times 7 = (40 \times 7) + (2 \times 7) = 280 + 14 = 294$$

You may wonder why the distributive law works in the first place. To understand it intuitively, imagine having 7 bags, each containing 42 coins, 40 of which arc gold and 2 of which are silver. How many coins do you have altogether? There are two ways to arrive at the answer. In the first place, by the very definition of multiplication, there are 42 x 7 coins. On the other hand, there are 40 x 7 gold coins and 2 x 7 silver coins. Hence, we have (40 x 7) + (2 x 7) coins altogether. By answering our question two ways, we have 42 x 7 = (40 x 7) + (2 x 7). Notice that the numbers 7, 40, and 2 could be replaced by any numbers (*a*, *b*, or *c*) and the same logic would apply. That's why the distributive law works!

Using similar reasoning with gold, silver, and platinum coins we can derive:

$$(b + c + d) x\, a = (bxa) + (c\, x\, a) + (d\, x\, a)$$

Hence, to do the problem 326 x 7, we break up 326 as 300 + 20 + 6, then distribute the 7, as follows: 326 x 7 = (300 + 20 + 6) x 7 = (300 x 7) + (20 x 7) + (6 x 7), which we then add up to get our answer.

As for squaring, the following algebra justifies my method. For any numbers *A and d*

$$A^2 = (A + d) \times (A - d) + d^2$$

Here, *A* is the number being squared; *d* can be any number, but I choose it to be the distance from *A* and the nearest multiple of 10. Hence, for (77), I set *d* = 3 and our formula tells us, 77^2= (77 + 3) x (77 - 3) + 3^2 = (80 x 74) + 9 = 5929.

The following algebraic relationship also works to explain my squaring method:

$$(z + d) = z^2 + 2zd + d^2 = z(z + 2d) + d^2$$

Hence, to square 41, we set z = 40 and $d = 1$ to get:

$$(41)^2 = (40 + 1)^2 = 40 \times (40 + 2) + 1^2 = 1681$$

Similarly,

$$(z-d)^2 = z(z-2d) + d^2$$

To find *if* when $z = 80$ and $d = 3$,

$$(77)^2 = (80 - 3)^2 = 80 \times (80 - 6) + 3^2 =$$
$$80 \times 74 + 9 = 5929$$

CHAPTER 4
MENTAL DIVISION

Mental division is a particularly handy skill to have, both in business and in daily life. How many times a week are you confronted with situations that call on you to divide things up evenly, such as a check at a restaurant? This same skill comes in handy when you want to figure out the cost per unit of a case of dog food on sale, or to split the pot in poker, or to figure out how many gallons of gas you can buy with a $20 bill. The ability to divide in your head can save you the inconvenience of having to pull out a calculator every time you need to compute something.

With mental division, the left-to-right method of calculation comes into its own. This is the same method we all learned in school, so you will be doing what comes naturally. I remember as a kid thinking that this left-to-right method of division is the way all arithmetic should be done. I have often speculated that if the schools could have figured out a way to teach division right-to-left, they probably would have done so!

1-DIGIT DIVISION

The first step when dividing mentally is to figure out how many digits will be in your answer. To see what I mean, try the following problem on for size:

$$179 \div 7$$

To solve $179 \div 7$, we're looking for a number, *Q*, such that 7 times *O* is 179. Now, since 179 lies between 7 x 10 = 70, and 7 x 100 = 700, *Q* must lie between 10 and 100, which means our answer must be a 2-digit number. Knowing that, we first determine the largest multiple of 10 that can be multiplied by 7 whose answer is below 179. We know that 7 x 20 = 140 and 7 x 30 = 210, so our answer must be in the 20's. At this point we can actually *say* the number 20 since that part of our answer will certainly not change. Next we subtract 179 - 140 = 39. Our problem has now been reduced to the division problem $39 \div 7$. Since 7 x 5 = 35, which is 4 away from 39, we have the rest of our answer, namely 5 with a remainder of 4, which we can now say. Altogether, we have said our answer, 25 with a remainder of 4, or if you prefer, $25\frac{4}{7}$. Here's what the process looks like:

$$\begin{array}{r} 25 \\ 7\overline{)179} \\ \underline{140} \\ 39 \\ \underline{35} \\ 4 \end{array} \leftarrow \textbf{remainder}$$

Answer: 25 with a remainder of 4, or $25\frac{4}{7}$

Let's try a similar division problem using the same method of mental computation:

$$675 \div 8$$

As before, since 675 falls between 8 x 10 = 80 and 8 x 100 = 800, your answer must be below 100 and therefore is a 2-digit number. To divide 8 into 675, notice that 8 x 80 = 640 and 8 x 90 = 720. Therefore, your answer is 80 something. But what is that "something"? To find out, subtract 640 from 675 for a remainder of 35. After saying the 80, our problem has now reduced to $35 \div 8$. Since 8 x 4 = 32, the final answer is 84 with a remainder of 3. We illustrate this problem as follows:

$$\begin{array}{r} 84 \\ 8\overline{)675} \\ \underline{640} \\ 35 \\ \underline{32} \\ 3 \end{array}$$

Answer: $84\frac{3}{8}$

Like most mental calculations, division can be thought of as a process of simplification. The more you calculate, the simpler the problem becomes. What began as $675 \div 8$ was simplified to a smaller problem $35 \div 8$.

Now let's try a division problem that results in a 3-digit answer:

$$947 \div 4$$

You can tell right away that your answer will have three digits because 947 falls between 4 x 100 = 400 and 4 x 1000 = 4000. Thus we must first find the largest multiple of 100 that can be *squeezed* into 947. Since 4 x 200 = 800, our answer is definitely in the 200's, so go ahead and say it! Subtracting 800 from 947 gives us our new division problem, $147 \div 4$. Since 4 x 30 = 120, we can now say the 30. After subtracting 120 from 147, we compute $27 \div 4$ to obtain the rest of the answer. Altogether, we have 236 with a remainder of 3, or $236\frac{3}{4}$.

$$
\begin{array}{r}
236 \\
4\overline{)947} \\
\underline{800} \\
147 \\
\underline{120} \\
27 \\
\underline{24} \\
3
\end{array}
$$

Answer: $236\frac{3}{4}$

The process is just as easy when dividing a 1-digit number into a 4-digit number, as in our next example.

$2196 \div 5$

You can see the answer will be in the hundreds because 2196 is between 5 x 100 = 500 and 5 x 1000 = 5000. After subtracting 5 x 400 = 2000, we can say the 400, and our problem has reduced to $196 \div 5$, which can be solved as in the previous examples.

$$
\begin{array}{r}
439 \\
5\overline{)2196} \\
\underline{2000} \\
196 \\
\underline{150} \\
46 \\
\underline{45} \\
1
\end{array}
$$

Answer: $439\frac{1}{5}$

Exercises: 1-Digit. Division

1) $9\overline{)318}$ 2) $5\overline{)726}$ 3) $7\overline{)428}$

4) $8\overline{)289}$ 5) $3\overline{)1328}$ 6) $4\overline{)2782}$

DECIMALIZATION

As you might guess, I like to work some magic when I convert fractions to decimals. In the case of 1-digit fractions, the best way is to commit the following fractions—from halves through elevenths—to memory. This isn't as hard as it sounds. As you'll see below, most 1-digit fractions have special properties that make them hard to forget. Anytime you can reduce a fraction to one you already know, you'll speed up the process.

Chances are you already know the decimal equivalent of the following fractions:

$\frac{1}{2} = .50$ $\frac{1}{3} = \overline{.333}$ $\frac{2}{3} = \overline{.666}$ (The bar signifies that the numbers will repeat indefinitely.)

Likewise:

$\frac{1}{4} = .25$ $\frac{2}{4} = \frac{1}{2} = .50$ $\frac{3}{4} = .75$

The fifths are easy to remember:

$\frac{1}{5} = .20$ $\frac{2}{5} = .40$ $\frac{3}{5} = .60$ $\frac{4}{5} = .80$

The 6ths require memorizing only two new answers:

$\frac{1}{6} = .1\overline{666}$ $\frac{2}{6} = \frac{1}{3} = \overline{.333}$ $\frac{3}{6} = \frac{1}{2} = .50$

$\frac{4}{6} = \frac{2}{3} = \overline{.666}$ $\frac{5}{6} = .8\overline{333}$

I'll return to the 7ths in a moment. The 8ths are a breeze:

$\frac{1}{8} = .125$ $\frac{2}{8} = \frac{1}{4} = .25$

$\frac{3}{8} = .375\ (3 \times \frac{1}{8} = 3 \times .125 = .375)$ $\frac{4}{8} = \frac{1}{2} = .50$

$\frac{5}{8} = .625\ (5 \times \frac{1}{8} = 5 \times .125 = .625)$ $\frac{6}{8} = \frac{3}{4} = .75$

$\frac{7}{8} = .875\ (7 \times \frac{1}{8} = 7 \times .125 = .875)$

The 9ths have a magic all their own:

$\frac{1}{9} = \overline{.111}$ $\frac{2}{9} = \overline{.222}$ $\frac{3}{9} = \overline{.333}$ $\frac{4}{9} = \overline{.444}$

$\frac{5}{9} = \overline{.555}$ $\frac{6}{9} = \overline{.666}$ $\frac{7}{9} = \overline{.777}$ $\frac{8}{9} = \overline{.888}$

The 10ths you already know:

$\frac{1}{10} = .10$ $\frac{2}{10} = .20$ $\frac{3}{10} = .30$

$\frac{4}{10} = .40$ $\frac{5}{10} = .50$ $\frac{6}{10} = .60$

$\frac{7}{10} = .70$ $\frac{8}{10} = .80$ $\frac{9}{10} = .90$

For the 11 ths, if you remember that $\frac{1}{11}$ = .0909, the rest is easy:

$\frac{1}{11} = \overline{.0909}$ $\frac{2}{11} = \overline{.1818}\ (2 \times \overline{.0909})$

$\frac{3}{11} = \overline{.2727}\ (3 \times \overline{.0909})$ $\frac{4}{11} = \overline{.3636}$ $\frac{5}{11} = \overline{.4545}$

$\frac{6}{11} = \overline{.5454}$ $\frac{7}{11} = \overline{.6363}$ $\frac{8}{11} = \overline{.7272}$

$\frac{9}{11} = \overline{.8181}$ $\frac{10}{11} = \overline{.9090}$

The 7ths are truly remarkable. Once you memorize $\frac{1}{7} = .\overline{142857}$, you can get all the other 7ths without having to compute them:

$\frac{1}{7} = .\overline{142857}$ $\frac{2}{7} = .\overline{285714}$ $\frac{3}{7} = .\overline{428571}$

$\frac{4}{7} = .\overline{571428}$ $\frac{5}{7} = .\overline{714285}$ $\frac{6}{7} = .\overline{857142}$

Note that the same pattern of numbers repeats itself in each fraction. Only the starting point varies. You can figure out the starting point in a flash by multiplying .14 by the numerator. In the case of $\frac{2}{7}$, $2 \times .14 = .28$, so use the sequence that begins with the 2, namely $.\overline{285714}$. Likewise with $\frac{3}{7}$, since 3 x .14 = .42, use the sequence that begins with 4, namely $.\overline{428571}$. Likewise the rest.

You will have to calculate fractions higher than $\frac{10}{11}$ as you would any other division problem. However, keep your eyes peeled for ways of simplifying such problems. For example, you can simplify the fraction $\frac{58}{16}$ by dividing both numbers by 2, to reduce it to $\frac{29}{8}$, which is easier to compute.

If the denominator of the fraction is an even number, you can simplify the fraction by reducing it in half, even if the numerator is odd. For example:

$$\frac{9}{14} = \frac{4.5}{7}$$

Dividing the numerator and denominator in half reduces it to a 7ths fraction. Although the 7ths sequence previously shown doesn't provide the decimal for $\frac{4.5}{7}$, once you begin the calculation, the number you memorized will pop up:

$$\begin{array}{r} .6\overline{428571} \\ 7\overline{)4.5000000} \\ \underline{4.2} \\ .3 \end{array}$$

As you can see, you needn't work out the entire problem. Once you've reduced it to dividing 3 by 7, you can make a great impression on an audience by rattling off this long string of numbers almost instantly!

Exercises: Decimalization

To solve the following problems, don't forget to employ the various 1-digit fractions you already know as decimals. Wherever appropriate, simplify the fraction before converting it to a decimal.

1) $\frac{2}{5}$	2) $\frac{4}{7}$	3) $\frac{3}{8}$

FRACTIONS

If you can manipulate whole numbers, then doing arithmetic with fractions is almost as easy. In this section, we review the basic methods for adding, subtracting, multiplying, dividing, and simplifying fractions. Those already familiar with fractions can skip this section with no loss of continuity.

Multiplying Fractions

To multiply two fractions, simply multiply the top numbers (called the *numerators),* then multiply the bottom numbers (called the *denominators).* For example,

$$\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$$

$$\frac{1}{2} \times \frac{5}{9} = \frac{5}{18}$$

What could be simpler! Try these exercises before going further.

Exercises: Multiplying Fractions

1) $\frac{3}{5} \times \frac{2}{7}$ 2) $\frac{4}{9} \times \frac{11}{9}$ 3) $\frac{6}{7} \times \frac{3}{4}$ 4) $\frac{9}{10} \times \frac{7}{8}$

Dividing Fractions

Dividing fractions is just as easy as multiplying fractions. There's just one extra step. First, turn the second fraction upside down (this is called the *reciprocal)* then multiply. For instance, the reciprocal of $\frac{4}{5}$ is $\frac{5}{4}$. Therefore,

$$\frac{2}{3} \div \frac{4}{5} = \frac{2}{3} \times \frac{5}{4} = \frac{10}{12}$$

$$\frac{1}{2} \div \frac{5}{9} = \frac{1}{2} \times \frac{9}{5} = \frac{9}{10}$$

Exercises

Now it's your turn. Divide these fractions,

1) $\frac{2}{5} \div \frac{1}{2}$

2) $\frac{1}{3} \div \frac{6}{5}$

3) $\frac{2}{5} \div \frac{3}{5}$

Simplifying Fractions

Fractions can be thought of as little division problems. For instance, $\frac{6}{3}$ **is** the same as $6 \div 3 = 2$. The fraction $\frac{1}{4}$ is the same as $1 \div 4$, (which is .25 in decimal form). Now we know that when we multiply any number by 1, the number stays the same. For example, $\frac{3}{5} = \frac{3}{5}$ x 1. But if we replace 1 with $\frac{2}{2}$, we get $\frac{3}{5} = \frac{3}{5} \times 1 = \frac{3}{5} \times \frac{2}{2} = \frac{6}{10}$ Hence, $\frac{3}{5} = \frac{6}{10}$. Likewise, if we replace 1 with $\frac{3}{3}$, we get $\frac{3}{5} = \frac{3}{5} \times \frac{3}{3} = \frac{9}{15}$. In other words, if we multiply the numerator and denominator by the same number, we get a fraction that is equal to the first fraction.

For another example,

$$\frac{2}{3} = \frac{2}{3} \times \frac{5}{5} = \frac{10}{15}$$

It is also true that if we divide the numerator and denominator by the same number, then we get a fraction that is equal to **the** first one.

For instance,

$$\frac{4}{6} = \frac{4}{6} \div \frac{2}{2} = \frac{2}{3}$$

$$\frac{25}{35} = \frac{25}{35} \div \frac{5}{5} = \frac{5}{7}$$

This is called *simplifying* the fraction.

Exercises

For the fractions below, can you find an equal fraction whose denominator is 12?

1) $\frac{1}{3}$ 2) $\frac{5}{6}$ 3) $\frac{3}{4}$ 4) $\frac{5}{2}$

Simplify these fractions.

5) $\frac{8}{10}$ 6) $\frac{6}{15}$ 7) $\frac{24}{36}$ 8) $\frac{20}{36}$

Adding Fractions

The easy case: Equal denominators. If the denominators are equal, then we add the numerators and keep the same denominators.

For instance,

$$\frac{3}{5}+\frac{1}{5}=\frac{4}{5}$$

$$\frac{4}{7}+\frac{2}{7}=\frac{6}{7}$$

Sometimes we can simplify our answer. For instance,

$$\frac{1}{8}+\frac{5}{8}=\frac{6}{8}=\frac{3}{4}$$

Exercises

1) $\frac{2}{9}+\frac{5}{9}$ 2) $\frac{5}{12}+\frac{4}{12}$ 3) $\frac{5}{18}+\frac{6}{18}$ 4) $\frac{3}{10}+\frac{3}{10}$

The trickier case: Unequal denominators. When the denominators arc not equal, then we replace our fractions with fractions where the denominators are equal.

For instance, to add $\frac{1}{3}+\frac{2}{15}$, we notice that $\frac{1}{3} = \frac{5}{15}$.

Therefore $\frac{1}{3} + \frac{2}{15} = \frac{5}{15} + \frac{2}{15} = \frac{7}{15}$.

To add $\frac{1}{2}+\frac{7}{8}$, we see that $\frac{1}{2} = \frac{4}{8}$, hence $\frac{1}{2} + \frac{7}{8} = \frac{4}{8} + \frac{7}{8} = \frac{11}{8}$.

Exercises

5) $\frac{1}{5} + \frac{1}{10}$

6) $\frac{2}{7} + \frac{5}{21}$

7) $\frac{1}{6} + \frac{5}{18}$

8) $\frac{3}{20} + \frac{1}{4}$

The problems in the last exercise were made easier because the smaller denominator divided into the larger one. But what if that's not the case? How would we add something like $\frac{1}{4}+\frac{2}{7}$? The trick is to create a *common denominator* by multiplying the top and bottom of each fraction by the *other* denominator. For example,

$$\frac{1}{4} = \frac{1}{4} \times \frac{7}{7} = \frac{7}{28}$$

$$\frac{2}{7} = \frac{2}{7} \times \frac{4}{4} = \frac{8}{28}$$

Thus $\frac{1}{4} + \frac{2}{7} = \frac{7}{28} + \frac{8}{28} = \frac{15}{28}$

How about $\frac{2}{3} + \frac{5}{8}$?

$$\frac{2}{3} = \frac{2}{3} \times \frac{8}{8} = \frac{16}{24}$$

$$\frac{5}{8} = \frac{5}{8} \times \frac{3}{3} = \frac{15}{24}$$

Thus $\frac{2}{3} + \frac{5}{8} = \frac{16}{24} + \frac{15}{24} = \frac{31}{24}$

Exercises

9) $\frac{1}{3} + \frac{1}{5}$

10) $\frac{2}{3} + \frac{3}{4}$

11) $\frac{3}{7} + \frac{3}{5}$

12) $\frac{2}{11} + \frac{5}{9}$

Subtracting Fractions

Subtracting fractions works very much like adding them. We illustrate with exampl and provide exercises for you to do.

$$\frac{3}{5} - \frac{1}{5} = \frac{2}{5}$$

$$\frac{4}{7} - \frac{2}{7} = \frac{2}{7}$$

$$\frac{5}{8} - \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

$$\frac{1}{3} - \frac{2}{15} = \frac{5}{15} - \frac{2}{15} = \frac{3}{15} = \frac{1}{5}$$

$$\frac{7}{8} - \frac{1}{2} = \frac{7}{8} - \frac{4}{8} = \frac{3}{8}$$

$$\frac{1}{2} - \frac{7}{8} = \frac{4}{8} - \frac{7}{8} = -\frac{3}{8}$$

$$\frac{2}{7} - \frac{1}{4} = \frac{8}{28} - \frac{7}{28} = \frac{1}{28}$$

$$\frac{2}{3} - \frac{5}{8} = \frac{16}{24} - \frac{15}{24} = \frac{1}{24}$$

Exercises

1) $\frac{8}{11} - \frac{3}{11}$

2) $\frac{12}{7} - \frac{8}{7}$

3) $\frac{13}{18} - \frac{5}{18}$

4) $\frac{4}{5} - \frac{1}{15}$

5) $\frac{9}{10} - \frac{3}{5}$

6) $\frac{3}{4} - \frac{2}{3}$

7) $\frac{7}{8} - \frac{1}{16}$

8) $\frac{4}{7} - \frac{2}{5}$

9) $\frac{8}{9} - \frac{1}{2}$

CHAPTER 5
THE ART OF "GUESSTIMATION"

So far, you have been perfecting the mental techniques necessary to figure out the exact answers to math problems. Often, however, all you will want is a ballpark estimate. Say you're getting quotes from different lenders on refinancing your home. All you really need at this information-gathering stage is a ballpark estimate of what your monthly payments will be. Or say you're settling a restaurant bill with a group of friends and you don't want to figure each person's bill to the penny. The guesstimation methods described in this chapter will make both these tasks—and many more just like them—much easier. Addition, subtraction, division, and multiplication all lend themselves to guesstimation. As usual, you will do your computations from left to right.

Addition Guesstimation

Guesstimation is a good way to make your life easier when the numbers of a problem are too long to remember. The trick is to round the original numbers up or down:

$$\begin{array}{r} 8{,}367 \\ +\ 5{,}819 \\ \hline 14{,}186 \end{array} \quad \approx \quad \begin{array}{r} 8{,}000 \\ +\ 6{,}000 \\ \hline 14{,}000 \end{array}$$

(≈ means approximately)

Notice that we rounded the first number down to the nearest thousand and the second number up. Since the exact answer is 14,186, our relative error is only $\frac{186}{14{,}186}$, or 1.3%.

If you want to be more exact, instead of rounding off to the nearest thousand, round off to the nearest hundred:

$$\begin{array}{r} 8{,}367 \\ +\ 5{,}819 \\ \hline 14{,}186 \end{array} \quad \approx \quad \begin{array}{r} 8{,}400 \\ +\ 5{,}800 \\ \hline 14{,}200 \end{array}$$

The answer is only 14 off from the exact answer, an error of less than .1%. This is what I call a good guesstimation!

Try a 5-digit addition problem, rounding to the nearest hundred:

$$\begin{array}{r} 46{,}187 \\ \underline{+\,19{,}378} \\ 65{,}565 \end{array} \quad \approx \quad \begin{array}{r} 46{,}200 \\ \underline{+\,19{,}400} \\ 65{,}600 \end{array}$$

By rounding to the nearest hundred our answer will always be off by less than 100. If the answer is larger than 10,000, your guesstimate will be within 1%.

Now let's try something wild:

$$\begin{array}{r} 23{,}859{,}379 \\ \underline{+7{,}426{,}087} \\ 31{,}285{,}466 \end{array} \quad \approx \quad \begin{array}{r} 24{,}000{,}000 \\ \underline{+7{,}000{,}000} \\ 31{,}000{,}000 \end{array} \quad \text{or} \quad \begin{array}{r} 23.9 \text{ million} \\ \underline{+7.4 \text{ million}} \\ 31.3 \text{ million} \end{array}$$

If you round to the nearest million, you get an answer of 31 million, off by roughly 285,000. Not bad, but you can do better by rounding to the nearest hundred thousand, as we've shown in the right-hand column. In this case you're off by only 15,000, which is awfully close when you're dealing with numbers to the third digit of the larger number (here the hundred-thousand digit), your guesstimate will always be within 1% of the precise answer. If you can compute these smaller problems exactly, you can guesstimate the answer to any addition problem.

Guesstimating at the Supermarket

Let's try a real-world example. Have you ever gone to the store and wondered what the total is going to be before the cashier rings it up? For estimating the total, my technique is to round the prices to the nearest 500. For example, while the cashier is adding the numbers shown below on the left, I mentally add the numbers shown on the right:

$1.39	$1.50
$0.87	$1.00
$2.46	$2.50
$0.61	$0.50
$3.29	$3.50
$2.99	$3.00
$0.20	$0.00
$1.17	$1.00
$0.65	$0.50
$2.93	$3.00
$3.19	$3.00
$19.75	$19.50

My final figure is usually within a dollar of the exact answer.

Subtraction Guesstimation

The way to guesstimate the answers to subtraction problems is the same—you round to the nearest thousand or hundreds digit, preferably the latter:

$$\begin{array}{r} 8{,}367 \\ \underline{-\,5{,}819} \\ 2{,}548 \end{array} \approx \begin{array}{r} 8{,}000 \\ \underline{-\,6{,}000} \\ 2{,}000 \end{array} \quad \text{or} \quad \begin{array}{r} 8{,}400 \\ \underline{-\,5{,}800} \\ 2{,}600 \end{array}$$

You can see that rounding to the nearest thousand leaves you with an answer quite a bit off the mark. By rounding to the second digit (hundreds, in the example), your answer will usually be within 3% of the exact answer. For this problem, your answer is off by only 52, a relative error of 2%. If you round to the third digit, the relative error will usually be below 1%. For instance:

$$\begin{array}{r} 439{,}412 \\ \underline{-\,24{,}926} \\ 414{,}486 \end{array} \approx \begin{array}{r} 440{,}000 \\ \underline{-\,20{,}000} \\ 420{,}000 \end{array} \quad \text{or} \quad \begin{array}{r} 439{,}000 \\ \underline{-\,25{,}000} \\ 414{,}000 \end{array}$$

By rounding the numbers to the third digit rather than to the second digit, you improve the accuracy of the estimate by a significant amount. The first guesstimate is off by about 1.3%, whereas the second guesstimate is only off by about 0.16%.

Division Guesstimation

The first step in guesstimating the answer to a division problem is 10 determine the magnitude of the answer:

$$\begin{array}{r} 9,645 \\ 6\overline{)57,870} \end{array} \approx \begin{array}{r} 9,000 \\ 6\overline{)58,000} \\ \underline{54,000} \\ 4,000 \end{array}$$

$$9\frac{2}{3}$$

The next step is to round off the larger numbers to the nearest thousand and change the 57,870 to 58,000. Dividing 6 into 58 is simple. The answer is 9 with a remainder. But the most important component in this problem is where to place the 9.

For example, multiplying 6 x 90 yields 540, while multiplying 6 x 900 yields 5400, both of which are too small. But 6 x 9000 = 54,000, which is pretty close to the answer. This tells you the answer is 9 thousand and something. You can estimate just what that something is by first subtracting 58 - 54 = 4. At this point you could bring down the 0 and divide 6 into 40, and so forth. But if you're on your toes you'll realize that dividing 6 into 4 gives you $\frac{4}{6}$ $\frac{2}{3}$ = .667. Since you know the answer is 9 thousand something, you're now in a position to guess 9667. In fact, the actual answer is 9645—darn close!

Here's an *astronomical* calculation for you. How many seconds does it take light to get from the Sun to the Earth? Well, light travels at 186,282 miles per second, and the sun is (on average) 92,960,130 miles away:

$$186,282\overline{)92,960,130}$$

1 doubt you're particularly eager to attempt this problem by hand. Fortunately, it's relatively simple to guesstimate an answer. First, simplify the problem:

$$\approx 186,000\overline{)93,000,000} = 186\overline{)93,000}$$

Now divide 186 into 930, which yields 5 with no remainder. Then tack back on the two zeros you removed from 93,000 and voilá—your answer is 500 seconds. The exact answer is 499.02 seconds, so this is a very respectable guesstimate.

Multiplication Guesstimation

You can use much the same techniques to guesstimate your answers to multiplication problems. For example,

$$\begin{array}{r} 88 \\ \underline{\times\ 54} \\ 4752 \end{array} \quad \approx \quad \begin{array}{r} 90 \\ \underline{\times\ 50} \\ 4500 \end{array}$$

Rounding up to the nearest multiple of 10 simplifies the problem considerably, but you're still off by 252, or about 5%. You can do better if you round both numbers by the same amount, but in opposite directions. That is, if you round 88 by *increasing 2,* you should also *decrease* 54 by 2:

$$\begin{array}{r} 88 \\ \underline{\times\ 54} \\ 4752 \end{array} \quad \approx \quad \begin{array}{r} 90 \\ \underline{\times\ 52} \\ 4680 \end{array}$$

Instead of a 1-by-l multiplication problem you now have a 2-by-l problem, which should be easy enough for you to do. Your guesstimation is off by only 1.5%.

When you guesstimate the answer to multiplication problems by rounding **the** larger number up and the smaller number down, your guesstimate will be a little low. If you round the larger number down and the smaller number up so that **the** numbers are closer together, your guesstimate will be a little high. The larger the amount by which you round up or down, the greater your guesstimate will be off from the exact answer. For example:

$$\begin{array}{r} 73 \\ \underline{\times\ 65} \\ 4745 \end{array} \quad \approx \quad \begin{array}{r} 70 \\ \underline{\times\ 68} \\ 4760 \end{array}$$

Since the numbers are closer together after you round them off, your guesstimate is a little high.

$$\begin{array}{r} 67 \\ \underline{\times\ 67} \\ 4489 \end{array} \quad \approx \quad \begin{array}{r} 70 \\ \underline{\times\ 64} \\ 4480 \end{array}$$

Since the numbers are farther apart, the estimated answer is too low, though again, not by much. You can see that this multiplication guesstimation method works quite well. Also notice that this problem is just 67' and that our approximation is just the first step of the squaring techniques. Let's look at one more example:

```
   83            85
x  52     ≈    x 50
 4316          4250
```

We observe that the approximation is most accurate when the original numbers are close together. Try estimating a 3 x 2 multiplication problem:

```
   728           731
x   63    ≈   x   60
45,864        43,860
```

By rounding 63 down to 60 and 728 up to 731, you create a 3-by-l multiplication problem, which puts your guesstimate within 2004 of the exact answer, an error of 4.3%.

Now try guesstimating the following 3-by-3 problem:

```
    367           359
x   492    ≈   x  500
180,564       179,500
```

You will notice that although you rounded both numbers up and down by 8, your guesstimate is off by over 1000. That's because the multiplication problem is larger and the size of the rounding number is larger, so the resulting estimate will be off by a greater amount. But the relative error is still under 1 %.

How high can you go with this system of guesstimating multiplication problems? As high as you want. You just need to know the names of large numbers. A thousand thousand is a million, and a thousand million is a billion. Knowing these names and numbers, try this one on for size:

```
28,657,493          29 million
x   13,864          x14 thousand
```

As before, the objective is to round the numbers to simpler numbers such as 29,000,000 and 14,000. Dropping the 0s for now, this is just a 2-by-2 multiplication problem: 29 x 14 = 406 (29 x 14 = 29 x 7 x 2 = 203 x 2 = 406). Hence the answer is roughly 406 billion, since a thousand million is a billion.

MOD SUMS (CASTING OUT 9'S)

Sometimes, when I do my math on paper, I check my answer by a method 1 call "mod sums" (because it is based on the elegant mathematics of modular arithmetic).

With the mod sums method, you sum the digits of each number until you are left with a single digit. For example, to compute the mod sum of 4328, add 4 + 3 + 2 + 8 = 17. Then add the digits of 17 to get 1 + 7 = 8. Hence the mod sum of 4328 is 8. For the following problem the mod sums of each number are computed as follows:

4328	→	17	→			8
884	→	20	→			2
620	→	8	→			8
1477	→	19	→	10	→	1
617	→	14	→			5
+725	→	14	→			+ 5
8651						29
↓						↓
20						11
↓						↓
(2)						(2)

As illustrated above, the next step is to add all the mod sums together (8 + 2 + 8 + 1 + 5 + 5). This yields 29, which sums to 11, which in turn sums to 2. Note that the mod sum of 8651, your original total of the original digits, is also 2. This is not a coincidence! If you've computed the answer and the mod sums correctly, your final mod sums must be the same. If they are different, you have definitely made a mistake somewhere. There is a 1 in 9 chance that the mod sums will match accidentally. If there is a mistake then this method will detect it 8 times out of 9.

The mod sum method is more commonly known to mathematicians and accountants as "casting out 9s" because the mod sum of a number happens to be equal to the remainder obtained when the number is divided by 9. In the case of the answer above—8651—the mod sum was 2. If you divide 8651 by 9 the answer is 961 with a remainder of 2. In other words, if you cast out 9 from 8651 a total of 961 times, you'll have a remainder of 2. There's one small exception to this. The sum of the digits of any multiple of 9 is also a multiple of 9. Thus, if a number is a multiple of 9 it will have a mod sum of 9, even though it has a remainder of 0.

You can also use mod sums to check your answers to subtraction problems. The key is to subtract the mod sums you arrive at and then compare that number to the mod sum of your answer.

65,717	→	8
- 38,491	→	- 7
27,226	→	(1)
↓		
(1)		

There's one extra twist. If the difference in the mod sums is a negative number or 0, add 9 to it. For instance:

$$\begin{array}{rcl}
42{,}689 & \rightarrow & 2 \\
\underline{-\ 18{,}764} & \rightarrow & \underline{-\ 8} \\
23{,}925 & & -6 + 9 = (3) \\
\downarrow & & \\
(3) & &
\end{array}$$

You can check your answers to multiplication problems with the mod sum method by multiplying the mod sums of the two numbers and computing the resulting number's mod sum. Compare this number to the mod sum of the answer. They should match. For example:

$$\begin{array}{rcr}
853 & \rightarrow & 7 \\
\underline{\times\ \ 762} & \rightarrow & \underline{\times\ 6} \\
649{,}986 & & 42 \\
\downarrow & & \downarrow \\
(6) & & (6)
\end{array}$$

If the mod sums don't match, you made a mistake. This method will detect mistakes, on average, 8 times out of 9.

GUESSTIMATION EXERCISES

Go through the following exercises for guesstimation math; then check your answers and computations with ours at the back of the book.

Exercises: Addition Guesstimation

Round these numbers up or down and see how close you can come to the exact answer.

1)	2)	3)	4)
1,479	57,293	312,025	8,971,011
+ 1,105	+ 37,421	+ 79,419	+ 4,016,367

Mentally estimate the total for the following column of price numbers by rounding to the nearest 500:

$ 2.67
$ 1.95
$ 7.35
$ 9.21
$ 0.49
$11.21
$ 0.12
$ 6.14
$ 8.31

Exercises: Subtraction Guesstimation
Estimate the following subtraction problems by rounding to the second or third digit.

1)	2)	3)	4)
4,926	67,221	526,978	8,349,241
+ 1,659	+ 9,874	+ 42,009	+ 6,103,839

Exercises: **Division Guesstimation**
Adjust the numbers in a way that allows you to guesstimate the following division problems.

1)	2)	3)	4)
$7\overline{)4379}$	$5\overline{)23{,}958}$	$289\overline{)5{,}102{,}357}$	$203{,}637\overline{)8{,}329{,}483}$

Exercises: **Multiplication** Guesstimation

Adjust the numbers in a way that allows you to guesstimate the following multiplication problems.

1) 98 x27	2) 76 x42	3) 88 x88	4) 539 x17
5) 312 x98	6) 639 x107	7) 428 x313	
8) 51,276 x 489	9) 104,972 x11,201	10) 5,462,741 x 203,413	

CHAPTER 6
INTRODUCTORY ALGEBRA AND SAT MATH

Playing with numbers has brought me great joy in life. I find that arithmetic can be just as entertaining as magic. But to understand the magical secrets of arithmetic requires algebra. Of course, there are other reasons to learn algebra (SAT's, modelling real world problems, computer programming, to name but a few), but what got me first interested in algebra was the desire to understand some mathematical magic tricks which I now present to you!

PSYCHIC MATH

Say to a volunteer in the audience, "Think of a number, any number, but to make it easy on yourself," you should say, "think of a 1-digit or 2-digit number." After you've reminded your volunteer that there's no way you could know his number, ask him to:

1. Double the number.

2. Add 12.

3. Divide the total by 2.

4. Subtract the original number.

Then say, "Was the answer you got by any chance the number 6?" Try this one on yourself first and you will see that the sequence always produces the number 6 no matter what number is originally selected.

For example, if the original number is 15.

1. $2 \times 15 = 30$

2. $30+12 = 42$

3. $42/2 = 21$

4. $21-15 = 6$

Why This Trick Works

Let us represent the original number by the letter y. Here are the functions you performed in the order you performed them:

1. $2y$

2. $2y+12$

3. $(2y+12)/2 = y + 6$

4. $y + 6 - y = 6$

So no matter what number your volunteer chooses, the final answer will always be 6. If you repeat this trick, have the volunteer add a different number at step 2 (say 18). The final answer will be half that number (2 x 15 = 30; 30 + **18** = 48; 48 + 2 = 24; 24 - 15 = 9).

THE MAGIC 1089!

Have your audience member take out a piece of paper and pencil and:

1. Secretly write down a 3-digit number in which the first digit is larger than the last digit.

2. Reverse that number and subtract it from the first number. If the result is 2-digit number, tell her to put a 0 in front of it.

3. Take that answer and add it to the reverse of itself.

At the end of this sequence your answer, 1089, will magically appear, no matter what number you originally chose. For example:

851	+ 685
- 158	- 586
693	099
+ 396	+ 990
1089	1089

Why This Trick Works

No matter what 3-digit number you or anyone else chooses in this game, the final result will always be 1089. Why? Let *abc* denote the unknown 3-digit number. Algebraically, this is equal to:

$$100a + 10b + c$$

When you reverse the number and subtract it from the original number you get the number *cba*, algebraically equal to:

$$\begin{array}{r} 100a + 10b + c \\ \underline{-(100c + 10b + a)} \\ 100(a - c) + (c - a) = 99(a - c) \end{array}$$

Hence, after subtracting in step 2, we must have one of the following multiples of 99: 099, 198, 297, 396, 495, 594, 693, 792, or 891, each one of which will produce 1089 after adding it to the reverse of itself, as we did in step 3.

LEAPFROG ADDITION

This trick combines a quick mental calculation with an astonishing prediction. Handing the spectator a card with ten lines, numbered 1 through 10, have the spectator think of two positive numbers between 1 and 20, and enter them on lines 1 and 2 of the card. Next have the spectator write the sum of lines 1 and 2 on line 3, then the sum of lines 2 and 3 on line 4, and so on as illustrated below.

1	9
2	2
3	11
4	13
5	24
6	37
7	61
8	98
9	159
10	257

Finally, have the spectator show you the card. At a glance, you can tell him the sum of all the numbers on the card. For instance, in our example, you could instantly announce that the numbers sum up to 671, faster than the spectator could do using a calculator! As a kicker, hand the spectator a calculator, and ask him to divide the number on line 10 by the number on line 9. In our example, the quotient $\frac{257}{159} = 1.616$. Have the spectator announce the first three digits of the quotient, then turn the card over. He 11 be surprised to see that you've already written the number 1.61!

Why This Trick Works

To perform the quick calculation, you simply multiply the number on line 7 by 11. Here 61 x 11 = 671. The reason this works is illustrated in the table below. If we denote the numbers on lines 1 and 2 by x and y respectively, then the sum of lines 1 through 10 must be 55x + 88y, which equals 11 times (5x + 8y), that is eleven times the number on line 7. As for the prediction, we exploit the fact that for any positive numbers, a, b, c, d, if $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. Thus the quotient of line 10 divided by line 9, $\frac{21x+34y}{13x+21y}$, must lie between $\frac{21x}{13x} = \frac{21}{13} \approx 1.615$ and $\frac{34y}{21y} = \frac{34}{21} \approx 1.619$. In fact, if you continue the leapfrog process indefinitely, the ratio of consecutive terms gets closer and closer to $\frac{1+\sqrt{5}}{2} \approx 1.618$, a number with so many beautiful and mysterious properties that it is often called the "golden ratio."

1	x
2	y
3	$x + y$
4	$x + 2y$
5	$2x + 3y$
6	$3x + 5y$
7	$5x + 8y$
8	$8x + 13y$
9	$13x + 21y$
10	$21x + 34y$
Total:	$55x + 88y$

MATHEMAGICS AND SAT MATH

Mastering Mathemagics may help students taking the SAT, an entrance exam required by many colleges. Even in its latest version, the mathematical reasoning section of the SAT still requires an understanding of basic arithmetic, but does not ask questions that require long tedious calculations. In fact, you are even allowed (and encouraged) to use a calculator during the exam. There is greater emphasis on mathematical *reasoning* rather than laborious computation. The Mathemagics user is encouraged to explore creative solutions to problems. <u>Look</u> at your numbers. Are there special features of the problem that you can exploit?

As an introduction, can you see a clever way to add,

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots\ldots + \left(\frac{1}{99} - \frac{1}{100}\right)$$

Don't let the parentheses fool you. Since addition is associative, we can rewrite the problem as,

$$1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} \quad \ldots\ldots \quad - \frac{1}{99} + \frac{1}{99} - \frac{1}{100}$$

Notice how the numbers will cancel each other out, and all we are left with is $1 - \frac{1}{100} = \frac{99}{100}$.

Here's one for you to try. Simplify:

$$\frac{1}{3} \times \frac{3}{5} \times \frac{5}{7} \times \frac{7}{9} \times \frac{9}{11} \times \frac{11}{13}$$

Notice how the 3's, 5's, 7's, 9's and 11 's cancel each other leaving us with $\frac{1}{13}$.

Although the SAT also covers such things as algebra and geometry, having a basic *number sense* is invaluable. Do your answers seem reasonable? With Mathemagics, we hope we have provided you with a greater number sense.

CHAPTER 1 ANSWERS

Answers: 2-Digit Addition

1)

$$\begin{array}{r} 23 \\ \underline{+16} \end{array} \xrightarrow{+10} \begin{array}{r} 33 \\ \underline{+6} \\ 39 \end{array}$$

2)

$$\begin{array}{r} 64 \\ \underline{+43} \end{array} \xrightarrow{+40} \begin{array}{r} 104 \\ \underline{+3} \\ 107 \end{array}$$

3)

$$\begin{array}{r} 95 \\ \underline{+32} \end{array} \xrightarrow{+30} \begin{array}{r} 125 \\ \underline{+2} \\ 127 \end{array}$$

4)

$$\begin{array}{r} 34 \\ \underline{+26} \end{array} \xrightarrow{+20} \begin{array}{r} 54 \\ \underline{+6} \\ 60 \end{array}$$

5)

$$\begin{array}{r} 89 \\ \underline{+78} \end{array} \xrightarrow{+70} \begin{array}{r} 159 \\ \underline{+8} \\ 167 \end{array}$$

6)

$$\begin{array}{r} 73 \\ \underline{+58} \end{array} \xrightarrow{+50} \begin{array}{r} 123 \\ \underline{+8} \\ 131 \end{array}$$

7)

$$\begin{array}{r} 47 \\ \underline{+36} \end{array} \xrightarrow{+30} \begin{array}{r} 77 \\ \underline{+6} \\ 83 \end{array}$$

8)

$$\begin{array}{r} 19 \\ \underline{+17} \end{array} \xrightarrow{+10} \begin{array}{r} 29 \\ \underline{+7} \\ 36 \end{array}$$

9)

$$\begin{array}{r} 55 \\ \underline{+49} \end{array} \xrightarrow{+40} \begin{array}{r} 95 \\ \underline{+9} \\ 104 \end{array}$$

10)

$$\begin{array}{r} 39 \\ \underline{+38} \end{array} \xrightarrow{+30} \begin{array}{r} 69 \\ \underline{+8} \\ 77 \end{array}$$

Answers: 3- Digit Addition

1)

$$\begin{array}{r} 242 \\ \underline{+137} \end{array} \xrightarrow{+100} \begin{array}{r} 342 \\ \underline{+37} \end{array} \xrightarrow{+30} \begin{array}{r} 372 \\ \underline{+7} \\ 379 \end{array}$$

2)

$$\begin{array}{r} 312 \\ \underline{+256} \end{array} \xrightarrow{+200} \begin{array}{r} 512 \\ \underline{+56} \end{array} \xrightarrow{+50} \begin{array}{r} 562 \\ \underline{+6} \\ 568 \end{array}$$

3)

$$\begin{array}{r} 635 \\ \underline{+\,814} \end{array} \xrightarrow{+\,800} \begin{array}{r} 1435 \\ \underline{+\,14} \end{array} \xrightarrow{+\,10} \begin{array}{r} 1445 \\ \underline{+\quad 4} \\ 1449 \end{array}$$

4)

$$\begin{array}{r} 457 \\ \underline{+\,241} \end{array} \xrightarrow{+\,200} \begin{array}{r} 657 \\ \underline{+\,41} \end{array} \xrightarrow{+\,40} \begin{array}{r} 697 \\ \underline{+\quad 1} \\ 698 \end{array}$$

5)

$$\begin{array}{r} 912 \\ \underline{+\,475} \end{array} \xrightarrow{+\,400} \begin{array}{r} 1312 \\ \underline{+\,75} \end{array} \xrightarrow{+\,70} \begin{array}{r} 1382 \\ \underline{+\quad 5} \\ 1387 \end{array}$$

6)

$$\begin{array}{r} 852 \\ \underline{+\,378} \end{array} \xrightarrow{+\,300} \begin{array}{r} 1152 \\ \underline{+\,78} \end{array} \xrightarrow{+\,70} \begin{array}{r} 1222 \\ \underline{+\quad 8} \\ 1230 \end{array}$$

7)

$$\begin{array}{r} 457 \\ \underline{+\,269} \end{array} \xrightarrow{+\,200} \begin{array}{r} 657 \\ \underline{+\,69} \end{array} \xrightarrow{+\,60} \begin{array}{r} 717 \\ \underline{+\quad 9} \\ 726 \end{array}$$

8)

$$\begin{array}{r} 878 \\ \underline{+\,797} \end{array} \xrightarrow{+\,700} \begin{array}{r} 1578 \\ \underline{+\,97} \end{array} \xrightarrow{+\,90} \begin{array}{r} 1668 \\ \underline{+\quad 7} \\ 1675 \end{array}$$

or

$$\begin{array}{r} 878 \\ \underline{+\,797} \end{array} \xrightarrow{+\,800} \begin{array}{r} 1678 \\ \underline{-\quad 3} \\ 1675 \end{array}$$

9)

$$\begin{array}{r} 276 \\ +\,689 \\ \hline \end{array} \xrightarrow{+\,600} \begin{array}{r} 876 \\ +\ \ 89 \\ \hline \end{array} \xrightarrow{+\,80} \begin{array}{r} 956 \\ +\ \ \ 9 \\ \hline 965 \end{array}$$

10)

$$\begin{array}{r} 877 \\ +\,539 \\ \hline \end{array} \xrightarrow{+\,500} \begin{array}{r} 1377 \\ +\ \ 39 \\ \hline \end{array} \xrightarrow{+\,30} \begin{array}{r} 1407 \\ +\ \ \ 9 \\ \hline 1416 \end{array}$$

11)

$$\begin{array}{r} 5400 \\ +\ \ 252 \\ \hline 5652 \end{array}$$

12)

$$\begin{array}{r} 1800 \\ +\,855 \\ \hline 2655 \end{array}$$

13)

$$\begin{array}{r} 6120 \\ +\,136 \\ \hline \end{array} \xrightarrow{+\,100} \begin{array}{r} 6220 \\ +\ \ 36 \\ \hline 6256 \end{array}$$

14)

$$\begin{array}{r} 7830 \\ +\,348 \\ \hline \end{array} \xrightarrow{+\,300} \begin{array}{r} 8130 \\ +\ \ 48 \\ \hline 8178 \end{array}$$

15)

$$\begin{array}{r} 4240 \\ +\,371 \\ \hline \end{array} \xrightarrow{+\,300} \begin{array}{r} 4540 \\ +\ \ 71 \\ \hline 4611 \end{array}$$

CHAPTER 2 ANSWERS

Answers: 2-Digit Subtraction

1)

38	- 20	18
- 23	→	- 3
		15

2)

84	- 60	24
- 59	→	+ 1
		25

3)

92	- 40	52
- 34	→	+ 6
		58

4)

67	- 50	17
- 48	→	+ 2
		19

5)

79	- 30	49
- 29	→	+ 1
		50

or

79	- 20	59
- 29	→	- 9
		50

6)

63	- 50	13
- 46	→	+ 4
		17

7)

51	- 30	21
- 27	→	+ 3
		24

8)

89	- 40	49
- 48	→	- 8
		41

9)

125	- 80	45
- 79	→	+ 1
		46

10)

148	- 90	58
- 86	→	+ 4
		62

Answers: 3-Digit Subtraction

1)

583 - 200 383 - 70 313
- 271 → - 71 → - 1
312

2)

936 - 700 236 - 20 216
- 725 → - 25 → - 5
211

3)

587 - 300 287
- 298 → + 2
289

4)

763 - 500 263
- 486 → + 14
277

5)

204 - 200 4
- 185 → + 15
19

6)

793 - 400 393
- 402 → - 2
391

7)

219 - 200 19
- 176 → + 24
43

8)

978 - 800 178
- 784 → + 16
194

9)

$$\begin{array}{r} 455 \\ \underline{-\ 319} \end{array} \xrightarrow{-400} \begin{array}{r} 55 \\ \underline{+\ 81} \\ 136 \end{array}$$

10)

$$\begin{array}{r} 772 \\ \underline{-\ 596} \end{array} \xrightarrow{-600} \begin{array}{r} 172 \\ \underline{+\ \ \ 4} \\ 176 \end{array}$$

11)

$$\begin{array}{r} 873 \\ \underline{-\ 357} \end{array} \xrightarrow{-400} \begin{array}{r} 473 \\ \underline{+\ 43} \\ 516 \end{array}$$

12)

$$\begin{array}{r} 564 \\ \underline{-\ 228} \end{array} \xrightarrow{-300} \begin{array}{r} 264 \\ \underline{+\ 72} \\ 336 \end{array}$$

13)

$$\begin{array}{r} 1428 \\ \underline{-\ 571} \end{array} \xrightarrow{-600} \begin{array}{r} 828 \\ \underline{+\ 29} \\ 857 \end{array}$$

14)

$$\begin{array}{r} 2345 \\ \underline{-\ 678} \end{array} \xrightarrow{-700} \begin{array}{r} 1645 \\ \underline{+\ 22} \\ 1667 \end{array}$$

15)

$$\begin{array}{r} 1776 \\ \underline{-\ 987} \end{array} \xrightarrow{-1000} \begin{array}{r} 776 \\ \underline{+\ 13} \\ 789 \end{array}$$

Answers: 2-by-l Multiplication

1)	2)	3)	4)
82	43	67	71
x 9	x 7	x 5	x 3
720	280	300	210
+18	+21	+35	+ 3
738	301	335	213

5)	6)			7)	8)
93	49		49	28	53
x 8	x 9		x 9	x 4	x 5
720	360	*or*	450	80	250
+24	+81		- 9	+32	+15
744	441		441	112	265

9)	10)	11)	12)
84	58	97	78
x 5	x 6	x 4	x 2
400	300	360	140
+20	+48	+28	+16
420	348	388	156

13)	14)	15)	16)
96	75	57	37
x 9	x 4	x 7	x 6
810	280	350	180
+54	+20	+49	+42
864	300	399	222

17)	18)	19)	20)
46	76	29	64
x 2	x8	x3	x8
80	560	60	480
+12	+48	+27	+32
92	608	87	512

Answers: 3-by-l Multiplication

1)	2)	3)	4)
431	637	862	957
x 6	x 5	x 4	x 6
2400	3000	3200	5400
+180	+150	+240	+300
2580	3150	3440	5700
+ 6	+ 35	+ 8	+ 42
2586	3185*	3448	5742

5)	6)	7)	8)
927	728	328	529
x 7	x 2	x 6	x 9
6300	1400	1800	4500
+140	+ 40	+120	+180
6440	1440	1920	4680
+ 49	+ 16	+ 48	+ 81
6489	1456	1968	4761

9)	10)	11)	12)
807	587	184	214
x 9	x 4	x 7	x 8
7200	2000	700	1600
+ 63	+ 320	+560	+ 80
7263	2320	1260	1680
	+ 28	+ 28	+ 32
	2348*	1288	1712

13)	14)	15)		
757	259	297	or	297
x 8	x 7	x 8		x 8
5600	1400	1600	300 x 8 =	2400
+400	+350	+720	- 3 x 8 =	- 24
6000	1750	2320		2376
+ 56	+ 63	+ 56		
6056	1813	2376		

16)	17)	18)	19)
751	457	339	134
x 9	x 7	x 8	x 8
6300	2800	2400	800
+450	+ 350	+240	+240
6750	3150	2640	1040
+ 9	+ 49	+ 72	+ 32
6759	3199	2712	1072

20)

611
x 3
1800
+ 33
1833

21)

578
x 9
4500
+630
5130
+ 72
5202

22)

247
x 5
1000
+200
1200
+ 35
1235*

23)

188
x 6
600
+480
1080
+ 48
1128

24)

968
x 6
5400
+360
5760
+ 48
5808

25)

499
x 9
3600
+810
4410
+ 81
4491

or

499
x 9
500 x 9 = 4500
-1 x 9 = - 9
4491

26)

670
x 4
2400
+280
2680

27)

429
x 3
1200
+ 60
1260
+ 27
1287

28)

862
x 5
4000
+300
4300
+ 10
4310*

29)

285
x 6
1200
+480
1680
+ 30
1710

30)

488
x 9
3600
+720
4320
+ 72
4392

31)

693
x 6
3600
+540
4140
+ 18
4158

32)

722
x 9
6300
+180
6480
+ 18
6498

33)

457
x 9
3600
+450
4050
+ 63
4113

34)

767
x 3
2100
+180
2280
+ 21
2301

35)

```
 312    or              312
x  9                   x  9
2700    500 x 9 =      2700
+ 90     -1 x 9 =     + 108
2790                   2808
+ 18
2808
```

36)

```
 691
 x 3
1800
+270
2070
+  3
2073
```

*With this kind of problem you can easily say the answer out loud as you go.

Answers: 2-Digit Squares

1)

14^2: $+4 \to 18$, $-4 \to 10$

$18 \times 10 = 180$

$180 + 16 = 196$ ($16 = 4^2$)

2)

27^2: $+3 \to 30$, $-3 \to 24$

$30 \times 24 = 720$

$720 + 9 = 729$ ($9 = 3^2$)

3)

65^2: $+5 \to 70$, $-5 \to 60$

$70 \times 60 = 4200$

$4200 + 25 = 4225$ ($25 = 5^2$)

4)

89^2: $+1 \to 90$, $-1 \to 88$

$90 \times 88 = 7920$

$7920 + 1 = 7921$ ($1 = 1^2$)

5)

98^2: $+2 \to 100$, $-2 \to 96$

$100 \times 96 = 9600$

$9600 + 4 = 9604$ ($4 = 2^2$)

6)

31^2: $+1 \to 32$, $-1 \to 30$

$32 \times 30 = 960$

$960 + 1 = 961$ ($1 = 1^2$)

7)

42

+1

41^2

-1

40

1680

$+\ 1 = 1^2$

1681

8)

60

+1

59^2

-1

58

3480

$+\ 1 = 1^2$

3481

9)

30

+4

26^2

-4

22

660

$+\ 16 = 4^2$

676

10)

56

+3

53^2

-3

50

2800

$+\ 9 = 3^2$

2809

11)

22

+1

21^2

-1

20

440

$+\ 1 = 1^2$

441

12)

68

+4

64^2

-4

60

4080

$+\ 16 = 4^2$

4096

13)

44

+2

42^2

-2

40

1760

$+\ 4 = 2^2$

1764

14)

60

+5

55^2

-5

50

3000

$+\ 25 = 5^2$

3025

15)

80

+5

75^2

-5

70

5600
+ 25 = 5^2
5625

16)

50

+5

45^2

-5

40

2000
+ 25 = 5^2
2025

17)

88

+4

84^2

-4

80

7040
+ 16 = 4^2
7056

18)

70

+3

67^2

-3

64

4480
+ 9 = 3^2
4489

19)

106

+3

103^2

-3

100

10,600
+ 9 = 3^2
10,609

20)

216

+8

208^2

-8

200

43,200
+ 64 = 8^2
43,264

CHAPTER 4 - DIVISION

Answers: 1-Digit Division

1)

$$\begin{array}{r} 35 \\ 9\overline{)318} \\ \underline{270} \\ 48 \\ \underline{45} \\ 3 \end{array}$$

2)

$$\begin{array}{r} 145 \\ 5\overline{)726} \\ \underline{500} \\ 226 \\ \underline{200} \\ 26 \\ \underline{25} \\ 1 \end{array}$$

3)

$$\begin{array}{r} 61 \\ 7\overline{)428} \\ \underline{420} \\ 8 \\ \underline{7} \\ 1 \end{array}$$

4)

$$\begin{array}{r} 36 \\ 8\overline{)289} \\ \underline{240} \\ 49 \\ \underline{48} \\ 1 \end{array}$$

5)

$$\begin{array}{r} 442 \\ 3\overline{)1328} \\ \underline{1200} \\ 128 \\ \underline{120} \\ 8 \\ \underline{6} \\ 2 \end{array}$$

6)

$$\begin{array}{r} 695 \\ 4\overline{)2782} \\ \underline{2400} \\ 382 \\ \underline{360} \\ 22 \\ \underline{20} \\ 2 \end{array}$$

Answers: Decimalization

1) $\frac{2}{5} = .4$

2) $\frac{4}{7} = .\overline{571428}$

3) $\frac{3}{8} = .375$

Answers: Multiplying Fractions

1) $\frac{3}{5} \times \frac{2}{7} = \frac{6}{35}$

2) $\frac{4}{9} \times \frac{11}{9} = \frac{44}{81}$

3) $\frac{6}{7} \times \frac{3}{4} = \frac{18}{28}$

4) $\frac{9}{10} \times \frac{7}{8} = \frac{63}{80}$

Answers: Dividing Fractions

1) $\frac{2}{5} \div \frac{1}{2} = \frac{4}{5}$

2) $\frac{1}{3} \div \frac{6}{5} = \frac{5}{18}$

3) $\frac{2}{5} \div \frac{3}{5} = \frac{10}{15}$

Answers: Simplifying Fractions

1) $\frac{1}{3} = \frac{4}{12}$

2) $\frac{5}{6} = \frac{10}{12}$

3) $\frac{3}{4} = \frac{9}{12}$

4) $\frac{5}{2} = \frac{30}{12}$

5) $\frac{8}{10} = \frac{4}{5}$

6) $\frac{6}{15} = \frac{2}{5}$

7) $\frac{24}{36} = \frac{2}{3}$

8) $\frac{20}{36} = \frac{5}{9}$

Answers: Adding Fractions

1) $\frac{2}{9} + \frac{5}{9} = \frac{7}{9}$

2) $\frac{5}{12} + \frac{4}{12} = \frac{9}{12} = \frac{3}{4}$

3) $\frac{5}{18} + \frac{6}{18} = \frac{11}{18}$

4) $\frac{3}{10} + \frac{3}{10} = \frac{6}{10} = \frac{3}{5}$

5) $\frac{1}{5} + \frac{1}{10} = \frac{2}{10} + \frac{1}{10} = \frac{3}{10}$

6) $\frac{2}{7} + \frac{5}{21} = \frac{6}{21} + \frac{5}{21} = \frac{11}{21}$

7) $\frac{1}{6} + \frac{5}{18} = \frac{3}{18} + \frac{5}{18} = \frac{8}{18} = \frac{4}{9}$

8) $\frac{3}{20} + \frac{1}{4} = \frac{3}{20} + \frac{5}{20} = \frac{8}{20} = \frac{2}{5}$

9) $\frac{1}{3} + \frac{1}{5} = \frac{5}{15} + \frac{3}{15} = \frac{8}{15}$

10) $\frac{2}{3} + \frac{3}{4} = \frac{8}{12} + \frac{9}{12} = \frac{17}{12}$

11) $\frac{3}{7} + \frac{3}{5} = \frac{15}{35} + \frac{21}{35} = \frac{36}{35}$

12) $\frac{2}{11} + \frac{5}{9} = \frac{18}{99} + \frac{55}{99} = \frac{73}{99}$

Answers: Subtracting Fractions

1) $\frac{8}{11} - \frac{3}{11} = \frac{5}{11}$

2) $\frac{12}{7} - \frac{8}{7} = \frac{4}{7}$

3) $\frac{13}{18} - \frac{5}{18} = \frac{8}{18} = \frac{4}{9}$

4) $\frac{4}{5} - \frac{1}{15} = \frac{12}{15} - \frac{1}{15} = \frac{11}{15}$

5) $\frac{9}{10} - \frac{3}{5} = \frac{9}{10} - \frac{6}{10} = \frac{3}{10}$

6) $\frac{3}{4} - \frac{2}{3} = \frac{9}{12} - \frac{8}{12} = \frac{1}{12}$

7) $\frac{7}{8} - \frac{1}{16} = \frac{14}{16} - \frac{1}{16} = \frac{13}{16}$

8) $\frac{4}{7} - \frac{2}{5} = \frac{20}{35} - \frac{14}{35} = \frac{6}{35}$

9) $\frac{8}{9} - \frac{1}{2} = \frac{16}{18} - \frac{9}{18} = \frac{7}{18}$

Answers: Addition Guesstimation

Exact:

1)	2)	3)	4)
1479	57,293	312,025	8,971,011
+1105	+37,421	+79,419	+4,016,367
2584	94,714	391,444	12,987,378

Guesstimate:

1)

1500	or	1480
+1100		+1100
2600		2580

2)

57,000	or	57,300
+37,000		+37,400
94,000		94,700

3)

310,000	or	312,000
+80,000		+79,000
390,000		391,000

4)

9 million	or	8.9 million	or	8.97 million
+4 million		+4.0 million		+4.02 million
13 million		12.9 million		12.99 million
		= 12,900,000		=12,990,000

5)

Exact:	Guesstimate:
\$ 2.67	\$ 2.50
\$ 1.95	\$ 2.00
\$ 7.35	\$ 7.50
\$ 9.21	\$ 9.00
\$ 0.49	\$ 0.50
\$11.21	\$11.00
\$ 0.12	\$ 0.00
\$ 6.14	\$ 6.00
\$ 8.31	\$ 8.50
\$47.45	\$47.00

Answers: Subtraction Guesstimation

Exact:

1)
$$\begin{array}{r} 4926 \\ \underline{-1659} \\ 3267 \end{array}$$

2)
$$\begin{array}{r} 67,221 \\ \underline{-9,874} \\ 57,347 \end{array}$$

3)
$$\begin{array}{r} 526,978 \\ \underline{-42,009} \\ 484,969 \end{array}$$

4)
$$\begin{array}{r} 8,349,241 \\ \underline{-6,103,839} \\ 2,245,402 \end{array}$$

Guesstimate:

1)
$$\begin{array}{r} 4900 \\ \underline{-1700} \\ 3200 \end{array}$$

2)
$$\begin{array}{r} 67,000 \\ \underline{-10,000} \\ 57,000 \end{array} \quad \text{or} \quad \begin{array}{r} 67,200 \\ \underline{-9,900} \\ 57,300 \end{array}$$

3)
$$\begin{array}{r} 530,000 \\ \underline{-40,000} \\ 490,000 \end{array} \quad \text{or} \quad \begin{array}{r} 527,000 \\ \underline{-42,000} \\ 485,000 \end{array}$$

4)
$$\begin{array}{r} 8.3 \text{ million} \\ \underline{-6.1 \text{ million}} \\ 2.2 \text{ million} \end{array} \quad \text{or} \quad \begin{array}{r} 8.35 \text{ million} \\ \underline{-6.10 \text{ million}} \\ 2.25 \text{ million} \end{array}$$

Answers: Division Guesstimation

Exact:

1)
$$\begin{array}{r} 625.57 \\ 7\overline{)4379} \end{array}$$

2)
$$\begin{array}{r} 4791.6 \\ 5\overline{)23,958} \end{array}$$

3)
$$\begin{array}{r} 17,655.21 \\ 289\overline{)5,102,357} \end{array}$$

4)
$$\begin{array}{r} 40.90 \\ 203,637\overline{)8,329,483} \end{array}$$

Guesstimate:

1)
$$\begin{array}{r} 630 \\ 7\overline{)4400} \end{array}$$

2)
$$\begin{array}{r} 4,800 \\ 5\overline{)24,000} \end{array}$$

3)
$$\begin{array}{r} 17,000 \\ 3\overline{)51,000} \end{array}$$

4)
$$\begin{array}{r} 40 \\ 200\overline{)8,000} \end{array}$$

Answers: Multiplication Guesstimation

Exact:

1)
98
x 27
2646

2)
76
x 42
3192

3)
88
x 88
7744

4)
539
x 17
9163

5)
312
x 98
30,576

6)
639
x 107
68,373

7)
428
x 313
133,964

8)
51,276
x 489
25,073,964

9)
104,972
x 11,201
1,175,791,372

10)
5,462,741
x 203,413
1,111,192,535,033

Guesstimate:

1)
100
x 25
2500

2)
78
x 40
3120

3)
90
x 86
7740

4)
540
x 17
9180

5)
310
x 100
31,000

6)
646
x 100
64,600

or

640
x 110
70,400

7)
430
x 310
133,300

8)
51,000
x 490
24,990,000

9)
105,000
x 11,000
1155 million = 1.155 billion

10)
5.5 million
x 200 thousand
1100 billion = 1.1 trillion